

ON DEGENERATE HYPERGEOMETRIC FUNCTIONS OF HUMBERT AND HORN

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Abstract

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MATHEMATICS

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ON DEGENERATE HYPERGEOMETRIC FUNCTIONS OF HUMBERT AND HORN

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In the present work we consider the confluent hypergeometric series Ξ_2 and H_3 ⁽¹⁾, which play an important role in the theory of the generalized wave equation ^(2,3):

$$Q[u] \equiv u_{\theta\theta} - u_{\sigma\sigma} - \frac{a}{\sigma}u_{\sigma} + c^2u = 0. \quad (1)$$

For them the following results are established:

1. Let $\beta > 0$, $\lambda > 0$, $\gamma = \delta - \beta + \mu - \lambda > 0$, $c_0\Gamma(\beta)\Gamma(\lambda)\Gamma(\gamma) = \Gamma(\delta)\Gamma(\mu)$.
Then

$$\begin{aligned} H_3(\alpha, \beta, \delta; x, y) = \\ = c_0 \int_0^1 \xi^{\beta-1} (1-\xi)^{\gamma-1} F(\mu-\lambda, \delta-\lambda, \gamma; 1-\xi) H_3(\alpha, \mu, \lambda; \xi x, y) d\xi. \end{aligned} \quad (2)$$

In particular, for $\lambda = \mu$,

$$H_3(\alpha, \mu, \mu; x, y) = (1-x)^{-\alpha} \bar{J}_{-\alpha}[2\sqrt{y(1-x)}],$$

and therefore (2) gives

$$H_3(\alpha, \beta, \delta; x, y) = c_1 \int_0^1 \xi^{\beta-1} (1-\xi)^{\delta-\beta-1} (1-\xi x)^{-\alpha} \bar{J}_{-\alpha}[2\sqrt{y(1-\xi x)}] d\xi, \quad (3)$$

where $\delta > \beta > 0$, $c_1\Gamma(\beta)\Gamma(\delta-\beta) = \Gamma(\delta)$. The inverse of (3) is the equality

$$(1-x)^{-\alpha} \bar{J}_{-\alpha}[2\sqrt{y(1-x)}] = c_2 \int_0^1 \xi^{\lambda-1} (1-\xi)^{\mu-\lambda-1} H_3(\alpha, \mu, \lambda; \xi x, y) d\xi, \quad (4)$$

which arises from (2) when $\delta = \beta$, $\mu > \lambda > 0$, $c_2 \Gamma(\lambda) \Gamma(\mu - \lambda) = \Gamma(\mu)$.

2. If $a > 0$, $b > 0$, $\gamma = \alpha + \beta - a - b > 0$, moreover α and $\beta \neq 0, -1, -2, \dots$, then

$$\begin{aligned} \Xi_2(a, b, c; x, y) = \\ = \mu_1 \int_0^1 \xi^{a-1} (1-\xi)^{\gamma-1} F(\alpha - b, \beta - b, \gamma; 1-\xi) \Xi_2(\alpha, \beta, c; x\xi, y) d\xi, \end{aligned} \quad (5)$$

where $\mu_1 \Gamma(a) \Gamma(b) \Gamma(\gamma) = \Gamma(\alpha) \Gamma(\beta)$. Conversely, putting $\gamma_2 > \gamma_1 \geq 0$, we find

$$\Xi_2(\alpha, \beta, \gamma_2; x, y_2) = \mu_2 \int_0^1 \Xi_2(\alpha, \beta, \gamma_1; xt, y_1 t) Q(y_1, y_2, t) dt, \quad (6)$$

$$\mu_2 \Gamma(\gamma_1) \Gamma(\gamma_2 - \gamma_1) = \Gamma(\gamma_2),$$

$$Q = t^{\gamma_1-1} (1-t)^{\gamma_2-\gamma_1-1} \bar{J}_{\gamma_2-\gamma_1-1}[2\sqrt{(y_1-y_2)(1-t)}].$$

In the case $y_1 = 0$, (6) gives an integral representation of the series Ξ_2 , and for $y_2 = 0$ its inversion ${}_2F_1 = U[\Xi_2]$ with respect to $F(\alpha, \beta, \gamma_1; x)$. Composing the considered integral operators, we arrive at a series of combined relations. For example, if we insert (4) (with $\alpha = \gamma_1 - \gamma_2 + 1$, $x = t$, $y = y_1 - y_2$) into (6), or substitute ${}_2F_1 = U[\Xi_2]$ into the known integral representations of the Appell functions $F_k(x, y)$ ($k = 1, 2, 3, 4$) (1), then as a result we arrive at relations connecting $\Xi_2(x, y)$ with $H_3(x, y)$ and $F_k(x, y)$. The integrals (2)–(6) generate a number of infinite expansions for H_3 and Ξ_2 . Thus, with the aid of (3) and Lommel's multiplication theorem for $\bar{J}_{-\alpha}[2\sqrt{y(1-x\xi)}]$, we find, in the interval $|x| < 1$,

$$H_3(\alpha, \beta, \delta; x, y) = \sum_{n=0}^{\infty} \frac{(\beta)_n (xy)^n}{n! (1-\alpha)_n (\delta)_n} F(\alpha, \beta + n, \delta + n; x) \bar{J}_{n-\alpha}(2\sqrt{y}). \quad (7)$$

Starting from (2) and the multiplication theorem for $H_3(\alpha, \mu, \lambda; \xi x, y)$, we arrive at the expansion of $H_3(\alpha, \beta, \delta; x, y)$ in terms of the functions $H_3(\alpha, \mu, \lambda - n; x, y)$, ($n = 0, 1, 2, \dots$). Finally, replacing in (6) $\bar{J}_{\nu}(2\sqrt{z})$ by the power series ${}_0F_1(\nu +$

$1, -z)$, we obtain an addition theorem for Ξ_2 in the argument y . As a result of the limiting transition

$$\lim_{\beta \rightarrow \infty} H_3(\alpha, \beta, \delta; x/\beta, y) = H_5(\alpha, \delta; x, y),$$

$$\lim_{\alpha \rightarrow \infty} \Xi_2(\alpha, \beta, \gamma; x/\alpha, y) = \Phi_3(\beta, \gamma; x, y),$$

the preceding formulas pass into analogous equalities for H_5 and Φ_3 , among which we note only

$$H_3(\alpha, \beta, \delta; x, y) = \frac{1}{\Gamma(\beta)} \int_0^\infty \xi^{\beta-1} e^{-\xi} H_5(\alpha, \delta; x\xi, y) d\xi \quad (\beta > 0). \quad (8)$$

Substituting (8), with $\delta = \beta$, into the known integral representations of the functions $\Phi_3(x, y)$, $\Psi_2(x, y)$, and $\Xi_2(x, y)$ (1), we connect these Humbert series with $H_5(x, y)$.

Let us now consider several improper integrals with Bessel functions reducible to H_3 , H_5 , and Ξ_2 .

I. Denote by $U_m(\beta, \mu, \nu)$ ($m = 0, 1, 2, \dots$) the expression

$$U_m = \int_0^\infty t^{\beta-2\nu-1} \bar{J}_\mu(a\sqrt{z^2+t^2}) J_{\nu+m}(bt) J_{\nu+m}(ct) dt$$

and put $a^2 > (b+c)^2$, $\nu > -1/2$, $-2m < \beta < \mu + 2\nu + 5/2$, $|\bar{\omega}|^2 = b^2 + c^2 - 2bc \cos \varphi$. Then we obtain

$$U_m = D_m \frac{(bc)^\nu}{a^\beta} \int_0^\pi H_3\left(\frac{\beta}{2} - \mu, \frac{\beta}{2}, \nu + 1; \frac{\bar{\omega}^2}{a^2}, \frac{a^2 z^2}{4}\right) C_m^\nu(\cos \varphi) \sin^{2\nu} \varphi d\varphi, \quad (9)$$

if

$$\pi \nu 2^{2-\beta} \Gamma(\mu + 1 - \beta/2) \Gamma(2\nu + m) D_m = m! \Gamma(\beta/2) \Gamma(\mu + 1).$$

In particular, when $\beta = 2(\nu + 1)$,

$$U_0[2(\nu + 1), \mu, \nu] = \frac{A_1(bc)^\nu}{a^{2\mu} R^{2(\nu-\mu+1)}} H_3\left(\nu - \mu + 1, \nu + \frac{1}{2}, 2\nu + 1; \frac{4bc}{R^2}, \frac{z^2 R^2}{4}\right), \quad (10a)$$

where $R = \sqrt{a^2 - (b - c)^2}$, $A_1 \Gamma(\nu + 1) \Gamma(\mu - \nu) = 2 \Gamma(\mu + 1)$. Conversely, assuming $(b - c)^2 < a^2 < (b + c)^2$, $A_2 \sqrt{\pi} \Gamma(\mu + 1/2) = \Gamma(\mu + 1)$, we arrive at the value

$$U_0[2(\nu + 1), \mu, \nu] = \frac{A_2 R^{2\mu-1}}{a^{2\mu} \sqrt{bc}} \Xi_2 \left(\frac{1}{2} + \nu, \frac{1}{2} - \nu, \mu + \frac{1}{2}; \frac{R^2}{4bc}, -\frac{z^2 R^2}{4} \right). \quad (10b)$$

II. The function Ξ_2 also arises in the study of the integral $V_m(\beta, \mu, \nu)$ of the form

$$V_m = \int_0^\infty t^{\beta-1} (z^2 + t^2)^m \bar{J}_{\mu+m} \left(a \sqrt{z^2 + t^2} \right) \bar{J}_{\mu+m} \left(b \sqrt{z^2 + t^2} \right) \bar{J}_\nu(ct) dt.$$

For it, putting $\omega = \sqrt{a^2 + b^2 - 2ab \cos \varphi}$, $c^2 > (a + b)^2$, $\mu > -1/2$, $0 < \beta < 2\mu + \nu + 5/2$, $m = 0, 1, 2, \dots$, we obtain

$$V_m = \frac{\delta_m}{(ab)^m c^\beta} \int_0^\pi \Xi_2 \left(\frac{\beta}{2} - \nu, \frac{\beta}{2}, \mu + 1; \frac{\omega^2}{c^2}, -\frac{z^2 \omega^2}{4} \right) C_m^\mu(\cos \varphi) \sin^{2\mu} \varphi d\varphi, \quad (11)$$

$$\pi \mu \Gamma(2\mu + m) \Gamma(\nu + 1 - \beta/2) \delta_m = 4^{\mu+m+\beta/2-1} m! \Gamma(\beta/2) \Gamma(\nu + 1) \Gamma^2(\mu + m + 1).$$

In particular, if $m = 0$, $b = 0$, $\bar{V}_0 = V_0|_{b=0}$, $0 < c < a$, $0 < \beta < \mu + \nu + 2$, (11) gives

$$\bar{V}_0 = \varphi(\beta, \nu) c^{-\beta} \Xi_2(\beta/2 - \nu, \beta/2, \mu + 1; a^2/c^2, -1/4 a^2 z^2), \quad (12a)$$

where $\varphi(\beta, \nu) = 2^{\beta-1} \Gamma(\beta/2) \Gamma(\nu + 1) / \Gamma(\nu + 1 - \beta/2)$. Conversely, from (9), for $0 < c < a$ it follows that

$$\bar{V}_0 = \varphi(\beta, \mu) a^{-\beta} H_3(\beta/2 - \mu, \beta/2, \nu + 1; c^2/a^2, 1/4 a^2 z^2). \quad (12b)$$

III. With the series H_5 there is associated the integral $W_m(\mu, \nu)$ ($m = 0, 1, 2, \dots$) of the form

$$W_m = \int_0^\infty t^{\mu-1} e^{-t^2} J_{\nu+m}(at) J_{\nu+m}(bt) {}_0F_2(\mu/2, \nu + \mu/2; -yt^2) dt, \quad (13)$$

namely, under the conditions $\nu > -1/2$, $2(\nu + m) + \mu > 0$, (13) reduces to

$$W_m = \gamma_m(ab)^\nu \int_0^\pi e^{-\omega^2/4} H_5\left(1 - \frac{\mu}{2}, \nu + 1; \frac{\omega^2}{4}, y\right) C_m^\nu(\cos \varphi) \sin^{2\nu} \varphi d\varphi.$$

Here

$$4\pi\nu\Gamma(2\nu + m)\gamma_m = m!\Gamma(\nu + \mu/2),$$

and, when $\mu = 2$,

$$W_m(2, \nu) = \frac{1}{2} \exp\left(-\frac{a^2 + b^2}{4}\right) J_0(2\sqrt{y}) J_{\nu+m}(1/2 ab).$$

In particular, $\lim_{b \rightarrow 0} [b^{-\nu} W_0]$ gives the integral representation

$$H_5 = \frac{e^x}{\Gamma(\delta - \alpha)} \int_0^\infty t^{\delta-\alpha-1} e^{-t} {}_0F_2(1 - \alpha, \delta - \alpha; -yt) \bar{J}_{\delta-1}(2\sqrt{xt}) dt,$$

from which, for $\delta > \alpha > 0$, $\alpha \neq 1, 2, \dots$, one obtains asymptotic estimates characterizing the behavior of the function H_5 as $x \rightarrow \infty$ and $y \rightarrow \infty$. With the aid of (10) and (12) one can construct a number of important solutions of equation (1). For example, denote by $H(\theta, \sigma, \theta_0, \sigma_0)$ and $\bar{H}(\theta, \sigma, \theta_0, \sigma_0)$ the Hadamard functions of two singular Tricomi problems considered in ⁽²⁾, and let $V(\beta) = (2\sigma_0)^{-\alpha} H$, $\bar{V}(\beta) = (2\sigma_0)^{-\alpha} \bar{H}$ ($\alpha = 2\beta$). Then, relying on (10a), we obtain:

$$V(\beta) = (4\sigma\sigma_0)^{1-\alpha} \bar{V}(1 - \beta), \quad \bar{V}(\beta) = \tilde{x} R^{-\alpha} H_3(\beta, \beta, 2\beta; \omega, \rho), \quad (14)$$

where $\omega = 4\sigma\sigma_0/R^2$, $\rho = 1/4 c^2 R^2$, $R = \sqrt{(\theta - \theta_0)^2 - (\sigma - \sigma_0)^2}$, $\tilde{x}\Gamma(1 - \beta)\Gamma(\alpha) = \Gamma(\beta)$. Since the functions (14) possess logarithmic singularities on the characteristics $\theta \pm \sigma = \theta_0 \pm \sigma_0$, they simultaneously give fundamental (elementary) solutions of equation (2). From (10a) there also arise integrals of equation (1)

$$V_1(\beta) = (\sigma\sigma_0)^{1-\alpha} \bar{V}_1(1 - \beta), \quad \bar{V}_1(\beta) = (\theta - \theta_0) R^{-\alpha-2} H_3(\beta + 1, \beta, 2\beta; \omega, \rho), \quad (15)$$

which have singularities of a different character on the lines $R = 0$. Finally, (10b) generates the functions $(R_1 = \sqrt{(\theta - \theta_0)^2 - (\sigma + \sigma_0)^2})$

$$U = (\sigma\sigma_0)^{-\beta} \Xi_2(\beta, 1 - \beta; 1; 1/\omega, -\rho),$$

$$U_1 = (\sigma\sigma_0)^{-\beta}\Xi_2(\beta, 1 - \beta, 1 - R_1^2/4\sigma\sigma_0, -1/4c^2R_1^2),$$

considered earlier in ^(2,3) (U_1 was denoted in ⁽²⁾ by u_5). Another important class of particular solutions of equation (1) arises from (12) ($t = \theta^2/\sigma^2$, $\xi = 1/4c^2\sigma^2$):

$$\begin{aligned} u_1^{(\nu)} &= \theta^\nu \Xi_2(-\nu/2, (1-\nu)/2, \beta + 1/2; 1/t, \xi), \\ u_2^{(\nu)} &= \theta^{-\nu} r^{2\nu-a} \Xi_2(\nu/2, (1+\nu)/2, \nu - \beta + 1, r^2/\theta^2, -c^2r^2/4), \\ u_3^{(\nu)} &= \sigma^\nu H_3((1-\nu-a)/2, -\nu/2, 1/2; t, -\xi), \\ u_4^{(\nu)} &= \sigma^\nu \sqrt{t} H_3((2-a-\nu)/2, (1-\nu)/2, 3/2; t, -\xi), \quad r = \sqrt{\theta^2 - \sigma^2}. \end{aligned} \quad (16a,b)$$

Since, together with $u(\theta, \sigma, a)$, equation (1) is also satisfied by $\bar{u} = \sigma^{1-a}u(\theta, \sigma, 2-a)$, in turn (16) generate four other integrals $\bar{u}_k^{(\nu)}$ ($k = 1, 2, 3, 4$). The functions $u_k^{(\nu)}, \bar{u}_k^{(\nu)}$, ($k = 3, 4$), as well as (10a), (12b), (14), and (15), are transformed by the formula

$$H_3 = (1-x)^{-\alpha} H_3[\alpha, \delta - \beta, \delta; x/(x-1), y(1-x)],$$

while (10a), (14), and (15) are brought to another form by means of the quadratic transformation

$$H_3(\alpha, \beta, 2\beta; x, y) = \left(\frac{2}{2-x}\right)^\alpha H_{10}\left[\alpha, \beta + 1/2; 1/4\left(\frac{x}{2-x}\right)^2, 1/2y(2-x)\right].$$

Let us note that the series

$$u = \sum_{n=0}^{\infty} [a_n u_1^{(n)} + \bar{a}_n \bar{u}_1^{(n)}],$$

composed of the functions (16a), solves the singular Cauchy problem

$$u(\theta, 0) = \sum_{n=0}^{\infty} a_n \theta^n, \quad u_\eta(\theta, 0) = \sum_{n=0}^{\infty} \bar{a}_n \theta^n,$$

and $u_4^{(0)}$ and $\bar{u}_4^{(0)}$ give the Riemann functions of this problem ⁽²⁾.

For $c = 0$, $V(\beta)$, $\bar{V}(\beta)$, u_k^ν , and $\bar{u}_k^\nu$ ($k = 1, 2, 3, 4$) turn into Hadamard functions and the well-known self-similar (homogeneous) solutions of the Euler–Poisson equation ^(4,5).

In the iteration method indicated in ⁽²⁾, $u_k^{(\nu)}$ and $\bar{u}_k^{(\nu)}$ play the role of the first approximation to analogous solutions of the Chaplygin equation $T[u] = 0$; moreover, as shown in ⁽²⁾, the subsequent approximations also belong to the class

considered here of degenerate hypergeometric functions of Horn and Humbert. Therefore the relations found for Ξ_2 and H_3 also give the corresponding properties of the principal integrals of the equations $Q[u] = 0$, $T[u] = 0$ listed above.

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Note: Figure translations are in progress. See original paper for figures.

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