

SOLUTION OF THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A WEDGE HAVING A FINITE CUT

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Abstract

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THEORY OF ELASTICITY

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SOLUTION OF THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A WEDGE HAVING A FINITE CUT

(Presented by Academician N. I. Muskhelishvili on 26 VII 1965)

Let an elastic body in the plane $\zeta = \xi + i\eta$ occupy the angle $-\alpha < \arg \zeta < \alpha$, $0 < \alpha \leq \pi$, which is cut along the bisector from the vertex of the angle; we shall take the length of the cut to be equal to unity.

Let the boundary $\arg \zeta = \pm\alpha$ be free of external stresses (this may be assumed without loss of generality), and let the stress components on the cut be prescribed:

$$Y_y = p_1(\xi), \quad X_y = q_1(\xi) \quad \text{on the upper edge of the cut,}$$

$$Y_y = p_2(\xi), \quad X_y = q_2(\xi) \quad \text{on the lower edge of the cut,} \quad (1)$$

where $p_1(\xi), p_2(\xi), q_1(\xi), q_2(\xi)$ are absolutely continuous functions.

This problem, in the case when $\alpha = \pi/2$ (a half-plane with a cut), was solved in work ⁽¹⁾. We propose another way of solving the problem.

Map the region occupied by the body onto the infinite strip $-1 \leq y \leq 1$, $-\infty < x < \infty$, cut along the negative half-axis $x < 0$, $y = 0$. The mapping is given by the function $\zeta = e^{\alpha z}$.

Consider two strips $G_1(-\infty < x < \infty, 0 < y < 1)$ and $G_2(-\infty < x < \infty, -1 < y < 0)$. By the method of N. I. Muskhelishvili ⁽²⁾, the problem formulated above is reduced to the problem of determining functions $\Phi_{01}(z), \Psi_{01}(z)$ and $\Phi_{02}(z), \Psi_{02}(z)$, holomorphic in the strips G_1 and G_2 , with the following boundary conditions:

$$\Phi_{01}(z_0) + \overline{\Psi_{01}(z_0)} = \alpha [Y_y^{(1)} - iX_y^{(1)}] e^{\alpha z_0}, \quad \text{Im } z_0 = 0,$$

$$\Phi_{01}(z_0) + \overline{\Psi_{01}(z_0)} + \frac{1}{\alpha} (e^{2i\alpha} - 1) \overline{\Phi_{01}(z_0)} = 0, \quad \text{Im } z_0 = 1; \quad (2)$$

$$\Phi_{02}(z_0) + \overline{\Psi_{02}(z_0)} = \alpha [Y_y^{(2)} - iX_y^{(2)}] e^{\alpha z_0}, \quad \text{Im } z_0 = 0;$$

$$\Phi_{02}(z) + \overline{\Psi_{02}(z_0)} + \frac{1}{\alpha} (e^{-2i\alpha} - 1) \overline{\Phi_{02}(z_0)} = 0, \quad \text{Im } z_0 = -1; \quad (3)$$

$$\Phi_{01}(z) - \Phi_{02}(z_0) = \Psi_{01}(z_0) - \Psi_{02}(z_0) = 0 \quad \text{for } \text{Im } z_0 = 0, \text{ Re } z_0 > 0; \quad (4)$$

$$Y_y^{(1)}(z_0) - Y_y^{(2)}(z_0) = X_y^{(1)}(z_0) - X_y^{(2)}(z_0) = 0 \quad \text{for } \text{Im } z_0 = 0, \text{ Re } z_0 > 0, \quad (5)$$

where

$$Y_y^{(k)}(z_0) = Y_y(e^{\alpha x} \cos \alpha y, e^{\alpha x} \sin \alpha y), \quad X_y^{(k)}(z_0) = X_y(e^{\alpha x} \cos \alpha y, e^{\alpha x} \sin \alpha y),$$

$$(x, y) \in G_k,$$

$$\Phi_{0k}(z) = \frac{d}{dz} \varphi(e^{\alpha z}), \quad \Psi_{0k}(z) = \frac{1}{\alpha} \Phi_{0k}(z) + \frac{d}{dz} \psi(e^{\alpha z}), \quad (x, y) \in G_k, \quad k = 1, 2.$$

$X_y^{(k)}, Y_y^{(k)}$ are the stress components in the strip G_k , and $\varphi(\zeta), \psi(\zeta)$ are the complex potentials for the wedge with a cut.

We shall seek the analytic functions $\Phi_{0k}(z), \Psi_{0k}(z)$, $z \in G$ ($k = 1, 2$), in the following form:

$$\Phi_{0k}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A_k(t)}{t} e^{-itz} dt - \frac{1}{\sqrt{2\pi}} \frac{d}{dz} \int_{-\infty}^{\infty} \frac{A_k(t)}{t} e^{-itz} dt - c_k, \quad z \in G_k, \quad (6)$$

$$\Psi_{0k}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{B_k(t)}{t} e^{-itz} dt - \frac{1}{\sqrt{2\pi}} \frac{d}{dz} \int_{-\infty}^{\infty} \frac{B_k(t)}{t} e^{-itz} dt + \bar{c}_k, \quad z \in G_k, \quad (7)$$

where

$$c_k = \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A_k(t)}{t} e^{-itz} dt;$$

at the point $t = 0$ the integrals are understood in the sense of the Cauchy principal value.

Below we shall show that the desired functions $A_k(t)$, $B_k(t)$ ($k = 1, 2$) are Fourier transforms of summable functions, continuous on the entire axis, except, possibly, at the point $t = 0$. In what follows the class of such functions will be denoted by R'_0 .

In the representations (6) and (7) passage to the limit is possible both under the sign of differentiation and under the sign of integration (as z tends to the boundary G_k), and differentiation may be performed any number of times under the integral sign when $\text{Im } z \neq 0$.

Let us introduce some further definitions and notation. Denote by R_0 the totality of all functions

$$\Omega(t) = \int_{-\infty}^{\infty} \omega(x) e^{itx} dx, \quad \text{where } \omega(x) \in L(-\infty, \infty);$$

R_0 is a certain ring of continuous functions on the closed line ⁽³⁾. Further, denote by R_0^+ (R_0^-) the subring of R_0 composed of functions

$$\Omega^+(t) = \int_0^{\infty} \omega(x) e^{itx} dx \quad \left(\Omega^-(t) = \int_{-\infty}^0 \omega(x) e^{itx} dx \right).$$

The ring obtained by extending R_0 (R_0^+ , R_0^-) by adjoining the identity to it will be denoted by R (R^+ , R^-).

Denote by $S_k(t)$, $T_k(t)$, $P_k(t)$, $Q_k(t)$ the Fourier transforms of the functions $ae^{\alpha x} Y_y^{(k)}(x, 0)$, $ae^{\alpha x} X_y^{(k)}(x, 0)$, $ae^{\alpha x} p_k(e^{\alpha x})$, $ae^{\alpha x} q_k(e^{\alpha x})$ ($k = 1, 2$).

Substituting the representations (6) and (7) into the boundary conditions (2) and (3), performing the Fourier transform and taking into account that $S_k(t) = \overline{S_k(-t)}$, $T_k(t) = \overline{T_k(-t)}$ ($k = 1, 2$), we obtain

$$A_1(t) = -\frac{t \left[2 \operatorname{sh} t e^{-t} - it \frac{e^{2i\alpha} - 1}{\alpha} \right]}{4 \left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] (1 + it)} S_1(t) + \frac{t \left[2 \operatorname{sh} t e^{-t} + it \frac{e^{2i\alpha} - 1}{\alpha} \right]}{4 \left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] (1 + it)} iT_1(t), \quad (8)$$

$$A_2(t) = \frac{t \left[2 \operatorname{sh} t e^t + it \frac{e^{-2i\alpha} - 1}{\alpha} \right]}{4 \left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] (1 + it)} S_2(t) - \frac{t \left[2 \operatorname{sh} t e^t - it \frac{e^{-2i\alpha} - 1}{\alpha} \right]}{4 \left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] (1 + it)} iT_2(t); \quad (9)$$

$$B_k(t) = \frac{t}{1 + it} [S_k(t) + iT_k(t)] + \bar{A}_k(-t) \quad (k = 1, 2). \quad (10)$$

By virtue of condition (4) we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{A_1(t) - A_2(t)}{t} e^{-itx} dt - \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{A_1(t) - A_2(t)}{t} e^{-itx} dt + c_2 - c_1 = 0$$

for $x > 0$,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{B_1(t) - B_2(t)}{t} e^{-itx} dt - \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} \frac{B_1(t) - B_2(t)}{t} e^{-itx} dt + \bar{c}_1 - \bar{c}_2$$

for $x > 0$;

whence we easily conclude that if $A_k(t), B_k(t) \in R_0$ ($k = 1, 2$), then $A_1(t) - A_2(t) \in R_0^-$, $B_1(t) - B_2(t) \in R_0^-$.

Applying condition (5), we obtain

$$T_1(t) = T^+(t) + Q_1^-(t), \quad T_2(t) = T^+(t) + Q_2^-(t); \quad (11)$$

$$S_1(t) = S^+(t) + P_1^-(t), \quad S_2(t) = S^+(t) + P_2^-(t), \quad (12)$$

where $S^+(t)$, $T^+(t)$ are the sought functions from the class R_0^+ .

From conditions (8)–(10) we shall have

$$\begin{aligned} A_1(t) - A_2(t) = & - \frac{t \left[\operatorname{sh} t \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha} t \right]}{\left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] (1 + it)} S^+(t) \\ & + \frac{t \left[\operatorname{sh} t \operatorname{ch} t - \frac{\sin 2\alpha}{2\alpha} t \right]}{\left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] (1 + it)} iT^+(t) + f_1(t); \end{aligned} \quad (13)$$

$$B_1(t) - B_2(t) = -\frac{t \left[\operatorname{sh} t \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha} t \right]}{\left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] (1 + it)} S^+(t) \\ - \frac{t \left[\operatorname{sh} t \operatorname{ch} t - \frac{\sin 2\alpha}{2\alpha} t \right]}{\left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] (1 + it)} iT^+(t) + f_2(t), \quad (14)$$

where $f_1(t)$, $f_2(t)$ are prescribed functions.

Adding equalities (12) and (13) and introducing the notation

$$\Phi^-(t) = [A_1(t) - A_2(t) + B_1(t) - B_2(t)] \sqrt{1 + it}; \quad (15)$$

$$\Phi^+(t) = -2S^+(t) \sqrt{1 - it}, \quad (16)$$

we obtain

$$\Phi^+(t) = \frac{\left[\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right] \sqrt{1 + t^2}}{(\operatorname{sh} t \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha} t) t} \Phi^-(t) + g(t). \quad (17)$$

It is easy to see that the free term $g(t) \in R'_0$, while the coefficient of the Hilbert boundary-value problem (17) belongs to the ring of functions R . Since the coefficient is positive on the entire axis $-\infty < t < \infty$, the index of the boundary-value problem (17) is $\chi = 0$.

On the basis of work [4], the Hilbert boundary-value problem (17) in the class of functions $\Phi^\pm(t) \in R_0^\pm$ has the unique solution

$$\Phi^\pm(t_0) = \frac{X^\pm(t_0)}{2} \left[\pm \frac{g(t_0)}{X^+(t_0)} + \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{g(t) dt}{X^+(t)(t - t_0)} \right], \quad (18)$$

where

$$X^\pm(t_0) = \exp \frac{1}{2} \left[\pm \ln \frac{\left(\operatorname{sh}^2 t_0 - \left(\frac{\sin \alpha}{\alpha} \right)^2 t_0^2 \right) \sqrt{1 + t_0^2}}{t_0 \left(\operatorname{sh} t \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha} t_0 \right)} \right. \\ \left. + \frac{1}{\pi i} \int_{-\infty}^{\infty} \ln \frac{\left(\operatorname{sh}^2 t - \left(\frac{\sin \alpha}{\alpha} \right)^2 t^2 \right) \sqrt{1 + t^2}}{t \left(\operatorname{sh} t \operatorname{ch} t + \frac{\sin 2\alpha}{2\alpha} t \right)} dt \right]. \quad (19)$$

From equality (16) we have

$$S^+(t) = -\frac{1}{2\sqrt{1-it}} \Phi^+(t).$$

Subtracting equality (14) from equality (13), we obtain a boundary-value problem analogous to problem (17), from which $T^+(t)$ is determined. From equalities (11)–(12) it follows that $S_k(t)\sqrt{1-it} \in R_0$, $T_k(t)\sqrt{1-it} \in R_0$ ($k = 1, 2$). Substituting the found $S_k(t)$, $T_k(t)$ ($k = 1, 2$) into (8)–(10), we determine $A_k(t)$, $B_k(t)$ ($k = 1, 2$). Thus, the problem is solved.

Now let us show that $A_k(t)$, $B_k(t) \in R'_0$. To this end we add to both sides of equalities (8) and (9), respectively, expressions of the form

$$\frac{te^{-t}}{(1+it)2\operatorname{ch}t} [S_1(t) - iT_1(t)], \quad -\frac{te^t}{(1+it)2\operatorname{ch}t} [S_2(t) - iT_2(t)]. \quad (20)$$

It is easy to see that the right-hand sides of the resulting equalities belong to the class R'_0 . In order to prove that $A_k(t)$, $B_k(t)$ belong to the class R'_0 , it is enough to show that the expressions (20) belong to the class R'_0 . This follows from the following lemma.

Lemma. Let $f(t+i\tau)$ be an analytic function in the strip $-h < \tau < h$, satisfying the conditions $|f(t+i\tau)| < A/|t|^\beta$, $|f'(t+i\tau)| < B/|t|^\gamma$ as $|t| \rightarrow \infty$, $\beta > 0$, $\gamma > 1$; then $f(t) \in R'_0$.

Proof. The function $f(s)e^{-isx}$, $s = t+i\tau$, for $x \leq 0$ is analytic in the strip $0 \leq \tau < h_1$ ($h_1 < h$) and vanishes at infinity. The same is true for $x \geq 0$ in the strip $-h_1 < \tau \leq 0$. Therefore, using Cauchy's theorem and the Sokhotski-Plemelj formulas, we obtain

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t} e^{-itx} dt = \begin{cases} e^{h_1 x} \Phi_1(x) + f(0)/2, \\ e^{-h_1 x} \Phi_2(x) - f(0)/2, \end{cases} \quad (21)$$

where

$$\Phi_1(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t+ih_1)}{t+ih_1} e^{-itx} dt, \quad \Phi_2(x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t-ih_1)}{t-ih_1} e^{-itx} dt. \quad (22)$$

According to the conditions of the lemma, $\Phi_1(x)$, $\Phi_2(x) \in H$ (H is the class of Hölder functions) and $\Phi_1(0) - \Phi_2(0) = f(0)$. Hence it follows that both parts of equality (21) satisfy the Hölder conditions. Let us show that $\Phi_1(x)$ and $\Phi_2(x)$ have everywhere a continuous derivative, except possibly at the point $x = 0$.

We rewrite the first expression in equality (22) as follows:

$$x\Phi_1(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{f(t+ih_1)}{t+ih_1} \right)' e^{-itx} dt,$$

whence, by differentiation, we obtain

$$x\Phi_1'(x) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} t \left(\frac{f(t+ih_1)}{t+ih_1} \right)' e^{-itx} dt - \Phi_1(x).$$

According to the conditions of the lemma, the right-hand side of the last equality belongs to the class H and is equal to zero for $x = 0$. Therefore $\Phi_1'(x)$ is a continuous function for $x \neq 0$ and $|\Phi_1'(x)| < c/|x|^\delta$, as $x \rightarrow 0$, $\delta < 1$. It is clear that $\Phi_2'(x)$ will have the same properties. Then

$$\mu(x) \equiv \frac{d}{dx} \left[\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t} e^{-itx} dt \right] \in L(-\infty, \infty),$$

whence, using the inversion formula, we obtain

$$f(t) = \int_{-\infty}^{\infty} \mu(x) e^{itx} dx.$$

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Note: Figure translations are in progress. See original paper for figures.

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