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TOPOLOGICAL EMBEDDINGS AND PSEUDOISOTOPY

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Abstract

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MATHEMATICS

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TOPOLOGICAL EMBEDDINGS AND PSEUDOISOTOPY

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1. We study here arbitrary topological embeddings, i.e. homeomorphic mappings, of a two-dimensional manifold M^2 into three-dimensional Euclidean space E^3 , and embeddings of a polyhedron P^k of dimension $k \leq n/2 - 1$ into Euclidean space E^n . A topological polyhedron $\mathcal{P}$ in E^n is called **tame** if there exists a homeomorphic mapping H of the space E^n onto itself such that $H(\mathcal{P})$ is a rectilinear polyhedron. Otherwise the polyhedron $\mathcal{P}$ is called **wild**. A polyhedron $\mathcal{P} \subset E^n$ is called **locally tame** at a point x if there exist a neighborhood U of the point x and a homeomorphic mapping $\varphi : U \rightarrow E^n$ such that $\varphi(U \cap \mathcal{P})$ is a rectilinear polyhedron. If for a point $x \in \mathcal{P}$ no such neighborhood exists, then x is called a **point of wildness** for $\mathcal{P}$. Bing ⁽²⁾ and Moise ⁽⁸⁾ showed that a locally tame polyhedron in E^3 is tame. If S^2 is a locally tame topological sphere in E^3 , then there exists a homeomorphic mapping $H : E^3 \rightarrow E^3$ carrying S^2 into a polyhedral sphere and, by Aleksandrov's theorem ⁽¹⁾, onto the boundary σ^2 of a three-dimensional simplex: $H(\sigma^2) = S^2$. The homeomorphism H can be chosen so that it does not change orientation and, consequently, it can be joined by an isotopy F_t , $0 \leq t \leq 1$, with the identity mapping of E^3 ⁽⁹⁾: $F_0 = 1$, $F_1 = H$. G. Gluck ⁽⁶⁾ proved that for a locally tame polyhedron $\mathcal{P}^k$ of dimension $k \leq n/2 - 1$ in E^n there exists an isotopy of E^n from the identity mapping carrying $\mathcal{P}^k$ into a rectilinear polyhedron. A. V. Chernavskii proved an analogous theorem for polyhedra of dimension $k < 2/3n - 1$ ⁽¹⁰⁾.

For wildly embedded polyhedra in E^n , by the very definition, there exists no homeomorphism of E^n onto itself carrying the given polyhedron into a rectilinear one.

The following questions arise:

- 1) Does there exist, for a topological polyhedron $\mathcal{P}^k$ in E^n , a not necessarily one-to-one mapping F of the space E^n onto itself and a rectilinear polyhedron P^k such that $F|_{P^k}$ is a homeomorphism and $F(P^k) = \mathcal{P}^k$?
- 2) Can the mapping F be chosen so that it is pseudoisotopic to the identity?

Lenning gave a positive answer to question 1) for an embedding of the two-dimensional sphere S^2 in E^3 ⁽¹¹⁾.

We give an affirmative answer to both questions for an embedding of a compact two-dimensional manifold in E^3 and for a topological polyhedron of dimension $k \leq 1/2 n - 1$ in E^n .

A **pseudoisotopy** is a homotopy F_t , $0 \leq t \leq 1$, which for every $t < 1$ is a homeomorphic mapping of the space onto itself. F_t is called an ε -**pseudoisotopy** if for any x, t and t' , $\rho[F_t(x)F_{t'}(x)] < \varepsilon$, where ρ is distance.

Theorem 1. *For an arbitrarily embedded two-dimensional sphere S^2 in E^3 there exists a pseudoisotopy from the identity mapping $F_t : E^3 \rightarrow E^3$, carrying the boundary σ^2 of a three-dimensional simplex into S^2 : $F_0 = 1$, $F_1(\sigma^2) = S^2$, such that $F_1|_{\sigma^2}$ and $F_1|_{E^3 \setminus F_1^{-1}(S^2)}$ are homeomorphisms and the set of points x whose inverse images $F_1^{-1}(x)$ are degenerate is zero-dimensional and is contained in the set of points of wildness of S^2 .*

Theorem 2. For an arbitrarily embedded two-dimensional compact manifold $M^2 \subset E^3$ and a number $\varepsilon > 0$, there exists a polyhedral manifold $\mathfrak{M}^2$ and an ε -pseudoisotopy F_t carrying $\mathfrak{M}^2$ onto M^2 and satisfying the conditions of Theorem 1.

Theorem 3. For an arbitrarily embedded in Euclidean space E^n polyhedron $\mathfrak{P}^k$ of dimension $k \leq n/2 - 1$ and a rectilinear polyhedron $P^k \subset E^n$, homeomorphic to $\mathfrak{P}^k$, there exists a pseudoisotopy Φ_t from the identity map carrying P^k onto $\mathfrak{P}^k$: $\Phi_0 = 1$, $\Phi(P^k) = \mathfrak{P}^k$, and moreover $\Phi_1|_{P^k}$ and $\Phi_1|_{E^n \setminus \Phi_1^{-1}(\mathfrak{P}^k)}$ are homeomorphisms, and the set of points x whose inverse images $\Phi_1^{-1}(x)$ are degenerate is contained in the set of points of wildness for $\mathfrak{P}^k$.

Corollary. For an arbitrarily embedded in E^n polyhedron $\mathfrak{P}^k$ of dimension $k \leq n/2 - 1$ and a number $\varepsilon > 0$, there exists a rectilinear polyhedron $P^k \subset E^n$ and an ε -pseudoisotopy carrying P^k onto $\mathfrak{P}^k$ and satisfying the conditions of Theorem 2.

We describe here the idea of the proof of Theorem 1. The proof of Theorem 2 is analogous. The proof of Theorem 3 is based on Gluck's Theorems 1.2 and 8.2 ⁽⁶⁾. The corollary to Theorem 3 follows from the proof of this theorem.

2. Idea of the proof of Theorem 1.

By Bing's approximation theorem, for every $\varepsilon > 0$ there exists a homeomorphic ε -shift h_ε of the sphere S^2 onto a polyhedral sphere Σ^2 . Let σ^2 be the boundary of a three-dimensional simplex lying in E^3 , and let f be a homeomorphic mapping of σ^2 onto S^2 , chosen so that for sufficiently small ε , $h_\varepsilon f : \sigma^2 \rightarrow \Sigma^2$ can be extended to an orientation-preserving homeomorphism $\varphi : E^3 \rightarrow E^3$. The isotopy F_t is constructed so that $F_1|_{\sigma^2} \equiv f$.

Choosing a decreasing sequence of numbers $\varepsilon_n \rightarrow 0$, we construct a sequence of tame spheres S_n converging to S^2 in such a way that the sphere S_{n-1} is carried by

an ε_{n-1} -isotopy $H_t^n : E^3 \rightarrow E^3$ onto S_n , and the sequence of homeomorphisms $\Phi_1^n = H_1^n H_1^{n-1} \dots H_1^1$, where $H_1^1(\sigma^2) = S_1$, converges uniformly to a continuous map $F_1 : E^3 \rightarrow E^3$ satisfying the conditions of the theorem.

By Bing's theorems ^(4,5), on the sphere S^2 one can choose an increasing sequence of tame Sierpiński curves* $X_1 \subset X_2 \subset \dots \subset X_n \subset \dots$ such that the diameters of the components of $S^2 \setminus X_n$ are less than $\varepsilon_n/2$, $X_{n+1} \setminus X_n$ lies in a finite number of components of $S^2 \setminus X_n$, and $\overline{X_{n+1} \setminus X_n}$ ** contains the boundaries of these components.

Let $\{\delta_{ni}\}$ be the open disks that are the components of $S^2 \setminus X_n$. If the sphere S^2 is wild, then such disks exist (Theorem 8.2 in ⁽⁵⁾), since X_n is tame. For each δ_{ni} we choose in E^3 a neighborhood u_{ni} such that

$$S^2 \cap u_{ni} = \delta_{ni}, \quad \bar{u}_{ni} \cap \bar{u}_{nj} = \Lambda, \quad i \neq j; \quad \text{diam } u_{ni} < 2 \text{ diam } \delta_{ni} < \varepsilon_n; \quad \bar{u}_{ni} \subset \bar{u}_{n-1,j},$$

if $\bar{\delta}_{ni} \subset \delta_{n-1,j}$; $u_{ni} = u_{n-1,j}$, if $\delta_{ni} = \delta_{n-1,j}$.

By Bing's theorem ⁽³⁾, for each disk δ_{ni} there exists a homeomorphic shift g_{ni} , fixed on the boundary $-\partial\delta_{ni}$, into a disk $\Delta_{ni} \subset u_{ni}$, locally polyhedral mod $\partial\Delta_{ni}$. We choose the homeomorphisms g_{ni} so that $g_{ni} = g_{n-1,j}$, if $\delta_{ni} = \delta_{n-1,j}$, and set

$$S_n = S^2 \setminus \bigcup \delta_{ni} \cup \bigcup \Delta_{ni}; \quad \Delta_{ni} \subset u_{ni}; \quad (1)$$

$$g_n : S^2 \rightarrow S_n; \quad g_n|_{\delta_{ni}} = g_{ni}, \quad g_n|_{S^2 \setminus \bigcup \delta_{ni}} = 1.$$

Each sphere S_n is locally tame mod X_n and, by Theorem 8.2 ⁽⁵⁾, is tame. Then $g_1 f : \sigma^2 \rightarrow S_1$, by the choice of f , can be extended to a homeo-

* A Sierpiński curve is a continuum X obtained from the sphere by deleting a countable everywhere dense set of open disks whose boundaries do not intersect. X is called tame if there exists a homeomorphism $\psi : E^3 \rightarrow E^3$ such that $\psi(X)$ lies in a plane.

** By $\bar{A}$ we denote the closure of the set A .

morphism $H_1^1 : E^3 \rightarrow E^3$ which does not change orientation, and, consequently, there exists an isotopy $H_t^1 : E^3 \rightarrow E^3$ such that

$$H_0^1 = 1; \quad H_1^1|_{\sigma^2} = g_1 f. \quad (2)$$

The isotopies $\Phi_t^n : E^3 \rightarrow E^3$ such that $\Phi_1^n|_{\sigma^2} = g_n f$ are constructed by induction. Φ_t^n is the product of the isotopies Φ_t^{n-1} and $\tilde{H}_t^n = H_t^n \Phi_1^{n-1}$, $\Phi_t^n = \Phi_t^{n-1} \circ \tilde{H}_t^n$, where H_t^n , $n \geq 2$, is an ε_{n-1} -isotopy and $H_1^n|_{S_{n-1}} = g_n g_{n-1}^{-1}$. By construction, the sphere S_{n+1} differs from S_n in a finite number of regions u_{ni} , $i = 1, 2, \dots, k_n$, and

$$g_{n+1} \Big|_{S^2 \setminus \bigcup_{i=1}^{k_n} \delta_{ni}} = g_n. \quad (3)$$

For each δ_{ni} , $n = 1, 2, \dots$; $i = 1, 2, \dots, k_n$, a disk $\delta'_{ni} \subset S^2$ is chosen so that $\partial\delta'_{ni} \subset X_n$ and $\partial\delta'_{ni}$ does not meet the boundaries of the components of $S^2 \setminus X_n$, and

$$\delta'_{ni} \supset \delta_{ni}; \quad \delta'_{ni} \cap \delta'_{nj} = \Lambda, \quad i \neq j;$$

$$\text{diam } \delta'_{ni} < (1 + \varepsilon_n) \text{diam } \delta_{ni};$$

for $k < n$: $\bar{\delta}'_{nj} \subset \delta'_{ki}$, if $\bar{\delta}_{nj} \subset \delta'_{ki}$; $\bar{\delta}'_{nj} \cap \bar{\delta}'_{ki} = \Lambda$, if $\bar{\delta}_{nj} \cap \bar{\delta}_{ki} = \Lambda$.

For each disk δ'_{ni} a neighborhood u'_{ni} is constructed satisfying the conditions:

$$u'_{ni} \cap S^2 = \delta'_{ni}; \quad \text{diam } u'_{ni} < 2 \text{diam } \delta_{ni} < \varepsilon_n;$$

$$\bar{u}_{nr} \subset u'_{ni}, \text{ if } \delta_{nr} \subset \delta'_{ni}; \quad \bar{u}'_{ni} \cap \bar{u}'_{kj} = \Lambda, \text{ if } \delta'_{ni} \cap \delta'_{kj} = \Lambda \quad (4)$$

$$\bar{u}'_{ni} \subset \bar{u}'_{kr}, \text{ if } k < n \text{ and } \delta_{ni} \subset \delta'_{kr}.$$

We construct the isotopy H_t^{n+1} , carrying S_n into S_{n+1} , so that

$$H_t^{n+1} \Big|_{E^3 \setminus \bigcup_{i=1}^{k_n} u'_{ni}} = 1, \quad 0 \leq t \leq 1.$$

The construction is carried out inside the regions u'_{ni} , and we omit the indices: $\Delta = g_n(\delta) \subset u \subset u'$. In $\Delta' = S_n \cap u'$ choose a disk $\bar{\Delta}$ with boundary in X_n : $\bar{\Delta} \subset \Delta \subset \Delta'$.

$\bar{\Delta}$ is tame, since S_n is tame, and one can construct an isotopy φ_t satisfying the conditions

$$\varphi_0 = 1; \quad \varphi_t|_{\partial u'} = 1; \quad \varphi_1(\bar{\Delta}) = \tilde{d},$$

where $\tilde{d}$ is a polyhedral disk. In view of (4) and (1), $S_{n+1} \cap u = g_{n+1}(S^2 \cap u) = D$; $S_{n+1} \cap u' = g_{n+1}(S^2 \cap u')$. $\varphi_1(\bar{\Delta} \setminus \Delta \cup D)$ is a tame disk lying in u' and locally polyhedral at the points of its boundary, since $\bar{\Delta} \setminus \Delta \cup D \subset S_{n+1}$.

Applying the theorems of Bing ⁽²⁾ and Kister ⁽⁷⁾, we construct an isotopy $\theta_t : u' \rightarrow u'$ such that

$$\theta_0 = 1; \quad \theta_t|_{\partial u' \cup \varphi_1(\Delta' \setminus \bar{\Delta})} = 1 \quad (5)$$

and $\widetilde{D} = \theta_1 \varphi_1(\bar{\Delta} \setminus \Delta \cup D)$ is a polyhedral disk.

Then a piecewise-linear isotopy $h_t : \bar{u}' \rightarrow \bar{u}'$ is constructed so that

$$h_0 = 1; \quad h_t|_{\partial u' \cup \varphi_1(\Delta' \setminus \widetilde{\Delta})} = 1; \quad h_1 \varphi_1(\widetilde{\Delta}) = \widetilde{D} \quad (6)$$

Denoting by φ_t^n , θ_t^n , and h_t^n the isotopies constructed as described in each u'_{ni} , $i = 1, 2, \dots, k_n$, and fixed on $E^3 \setminus \bigcup_{i=1}^{k_n} u'_{ni}$, we put

$$\Gamma^{n+1} = \varphi_t^n \circ \tilde{h}_t^n \circ \tilde{\theta}_t^n \circ \varphi_t^n,$$

where

$$\tilde{h}_t^n = h_t^n \varphi_1^n; \quad \tilde{\theta}_t^n = (\theta_t^n)^{-1} \tilde{h}_1^n; \quad \tilde{\varphi}_t^n = (\varphi_t^n)^{-1} \tilde{\theta}_1^n.$$

Γ_1^{n+1} carries each disk $\widetilde{\Delta}_{ni}$ into the disk $\widetilde{\Delta}_{ni} \setminus \Delta_{ni} \cup D_{ni}$ and, by (5), (6), $\Gamma_1^{n+1}|_{S_n \setminus \bigcup \widetilde{\Delta}_{ni}} = 1$, hence $\Gamma_1^{n+1}(S_n) = S_{n+1}$.

The homeomorphic mapping

$$\psi^{n+1} = g_{n+1} g_n^{-1} (\Gamma_1^{n+1})^{-1} : S_{n+1} \rightarrow S_{n+1}$$

is fixed on

$$S_{n+1} \setminus \bigcup_{i=1}^{k_n} (\widetilde{\Delta}_{ni} \setminus \Delta_{ni} \cup D_{ni}),$$

therefore there exists an isotopy $\Psi^{n+1} : E^3 \rightarrow E^3$ such that

$$\Psi_0^{n+1} = 1; \quad \Psi_t^{n+1} \Big|_{E^3 \setminus \bigcup_{i=1}^{k_n} u'_{ni}} = 1; \quad \Psi_t^{n+1} \Big|_{S_{n+1}} = \psi^{n+1}.$$

Put

$$H_t^{n+1} = \Gamma_t^{n+1} \circ (\Psi_t^{n+1} \Gamma_1^{n+1}); \quad \widetilde{H}_t^{n+1} = H_t^{n+1} \Phi_1^n.$$

From (2), by induction we find

$$\Phi_1^{n+1} \Big|_{\sigma^2} = g_{n+1} f. \quad (7)$$

3. The sequence of homeomorphisms $\Phi_1^n : E^3 \rightarrow E^3$ converges uniformly to a continuous mapping $F : E^3 \rightarrow E^3$. Since Φ_1^n is joined to Φ_1^{n+1} by the ε_n -isotopy $\widetilde{H}_t^{n+1}$, then, after making for each isotopy $\widetilde{H}_t^n$ a linear change of the variable t , $\chi_n : [0, 1] \rightarrow [1 - 1/2^{n-1}, 1 - 1/2^n]$, and putting

$$F_0 = H_0^1 = 1; \quad F_t = \widetilde{H}_{\chi_n^{-1}(t)}, \quad 1 - 1/2^{n-1} \leq t \leq 1 - 1/2^n; \quad F_1 \equiv F = \lim_{n \rightarrow \infty} \Phi_1^n,$$

we obtain a pseudo-isotopy $F_t : E^3 \rightarrow E^3$. The relation $F_1|_{\sigma^2} \equiv f$ follows from (7).

It is easily proved that the set K of points whose inverse images are degenerate is, for any n , contained in

$$V_n = \bigcup_{k=n}^{\infty} \bigcup_{i=1}^{n_k} u_{ki}.$$

By (4), V_n is the sum of pairwise nonintersecting regions of diameter $< \varepsilon_n$, and $u_{ki} \cap S^2 \neq \Lambda$. Therefore $K \subset S^2$; $\dim K = 0$. Since $u_{ki} \cap S^2$ is a wild disk, K is contained in the set of wildness points of S^2 , and the conditions of Theorem 1 are satisfied.

Let us note that, for an embedding in E^3 of the one-dimensional sphere S^1 , a theorem analogous to Theorem 1 is impossible, since tame spheres Σ^1 approximating the wild sphere S^1 may, when approaching S^1 , form increasingly complicated knots.

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