

# THE CHARACTERISTIC EQUATION OF THE THEORY OF RADIATION TRANSFER

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1966

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**Abstract**

**Full Text**

UDC 517.947.32

**MATHEMATICS**

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## **THE CHARACTERISTIC EQUATION OF THE THEORY OF RADIATION TRANSFER**

*(Presented by Academician M. V. Keldysh on 16 IX 1965)*

The problem of determining the asymptotics of the spatial-angular distribution of radiation in the depth of a thick plane layer of matter reduces to the study of the equation <sup>(1)</sup>

$$(1 + k\vec{\omega}\mathbf{n})\Phi(\vec{\omega}) = \hat{g}\Phi(\vec{\omega}). \quad (1)$$

Here  $k$  is a parameter,  $\vec{\omega}$  is a vector running over the unit sphere  $\Omega$  of three-dimensional Euclidean space;  $\mathbf{n}$  is a fixed vector from  $\Omega$ ;  $\hat{g}$  is an integral operator:

$$\hat{g}f(\vec{\omega}) = \int_{\Omega} g(\vec{\omega}\vec{\omega}')f(\vec{\omega}') d\omega'.$$

The kernel  $g(\vec{\omega}\vec{\omega}')$  is determined by specifying the scattering indicatrix  $g(\mu)$ ,  $\mu \in [-1, 1]$ , and depends on the scalar product  $\vec{\omega}\vec{\omega}'$  of the vectors  $\vec{\omega}$  and  $\vec{\omega}'$  from  $\Omega$ ;

$$\int_{\Omega} \dots d\omega'$$

is the integral with respect to Lebesgue measure on  $\Omega$ .

We shall be interested in the question of for which values of  $k$  there exist, and how the nontrivial solutions of (1) are arranged; and how the solutions of the nonhomogeneous equation corresponding to (1) depend on  $k$ .

We shall assume that

$$1) \quad g \in L_2(\Omega); \quad g(\mu) \geq 0 \text{ on } [-1, 1]; \quad 0 < 2\pi \int_{-1}^1 g(\mu) d\mu \leq 1.$$

This condition makes it possible to expand  $g(\mu)$  in a Legendre-polynomial series convergent in the mean:

$$g(\mu) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} g_n P_n(\mu), \quad 0 < g_0 \leq 1,$$

$$|g_n| < g_0 \quad \text{for } n = 1, 2, \dots$$

Let  $L_2(\Omega)$  and  $C(\Omega)$  be the spaces of functions on  $\Omega$  that are respectively square-summable and continuous;

$$(f_1, f_2) = \int_{\Omega} f_1(\vec{\omega}) \overline{f_2(\vec{\omega})} d\omega$$

is the scalar product in  $L_2(\Omega)$ ;

$$\|f\| = (f, f)^{1/2}, \quad f \in L_2(\Omega),$$

and

$$\|f\|_{C(\Omega)} = \sup_{\vec{\omega} \in \Omega} |f(\vec{\omega})|, \quad f \in C(\Omega).$$

Denote by  $\vartheta$  the zero of  $L_2(\Omega)$ , and by  $Z_0$  the complex  $k$ -plane with cuts along the real axis from  $-\infty$  to  $-1$  and from  $1$  to  $\infty$ .

**Theorem 1.**  $\hat{g}$  is a linear bounded self-adjoint operator mapping  $L_2(\Omega)$  into itself, with norm  $\|\hat{g}\| = g_0$ .  $\hat{g}(L_2(\Omega)) \subset C(\Omega)$ ; for  $f \in L_2(\Omega)$

$$\|\hat{g}f\|_{C(\Omega)} \leq \|f\| \left( 2\pi \int_{-1}^1 g^2(\mu) d\mu \right)^{1/2};$$

as an operator acting from  $L_2(\Omega)$  to  $C(\Omega)$ ,  $\hat{g}$  is completely continuous.

Define, for  $k \in Z_0$ ,

$$\hat{U}(k)f(\vec{\omega}) = (1 + k\vec{\omega}\mathbf{n})^{-1} \hat{g}f(\vec{\omega}).$$

$$\hat{U}(k) : L_2(\Omega) \rightarrow$$

$L_2(\Omega)$ ,  $\hat{U}(k)$  is completely continuous. Let  $\sigma(\hat{U}(k))$  be its spectrum. Obviously, (1) is solvable if and only if  $1 \in \sigma(\hat{U}(k))$ .

It can be shown that for  $k \in (-1, 1)$  and sufficiently large  $p$

$$\inf_{\vec{\omega} \in \Omega} \hat{U}^p(k) f(\vec{\omega}) > 0$$

for every nonnegative  $f(\vec{\omega})$ . Relying on the theory of cones in Banach spaces <sup>(2)</sup>, from this we obtain:

**Theorem 2.** Let  $k \in (-1, 1)$ ,  $M(k) = \sup\{|\lambda| \mid \lambda \in \sigma(\hat{U}(k))\}$ . Then:

- I.  $M(k) > 0$ ,  $M(k) \in \sigma(\hat{U}(k))$ .
- II. If  $\lambda \in \sigma(\hat{U}(k))$ ,  $\lambda \neq M(k)$ , then  $|\lambda| < M(k)$ .
- III.  $M(k)$  is a simple eigenvalue of  $\hat{U}(k)$ .
- IV. There exists exactly one function  $\Phi_k \in L_2(\Omega)$  possessing the following properties:

$$\hat{U}(k)\Phi_k = M(k)\Phi_k, \quad (\Phi_k, 1) = 4\pi.$$

Moreover,

V.

$$\inf_{\vec{\omega} \in \Omega} \Phi_k(\vec{\omega}) > 0.$$

- VI. If  $x \neq 0$ ,  $x \in C(\Omega)$ ,  $x(\vec{\omega}) \geq 0$  on  $\Omega$ , and  $x$  is an eigenfunction of the operator  $\hat{U}(k)$ , then there exists  $a > 0$  such that  $x = a\Phi_k$ .

In order to prove the existence of nontrivial solutions of (1), it is now sufficient to establish the solvability of the equation  $M(k) = 1$  in the interval  $-1 < k < 1$ .

Let us note that  $\hat{U}(k)$  is an analytic function for  $k \in Z_0$ , and for  $k_0 \in (-1, 1)$ , according to Theorem 2,  $M(k_0)$  is a simple isolated eigenvalue of  $\hat{U}(k_0)$ . This permits one to apply to the analysis of the behavior of  $M(k)$  and  $\Phi_k$  in a neighborhood of the point  $k_0$  the theory of analytic perturbations of operators (for example <sup>(6)</sup>), which gives:

**Theorem 3.** There exist an open set  $G_0 \subset Z_0$ , a numerical function  $\lambda(k)$ , and a function  $e(k)$  with values in  $L_2(\Omega)$ , defined on  $G_0$ , such that:

- I.  $(-1, 1) \subset G_0$ .
- II.  $\lambda(k)$  and  $e(k)$  are holomorphic in  $G_0$ .
- III. For  $k \in G_0$ ,  $\|e(k)\| > 0$ ,  $\lambda(k) \neq 0$ ,  $\hat{U}(k)e(k) = \lambda(k)e(k)$ .
- IV. For  $k \in (-1, 1)$ ,  $\lambda(k) = M(k)$ ,  $e(k) = \Phi_k$ .

According to Theorem 3, in a neighborhood of each point  $k_0 \in G_0$ ,  $e(k)$  is expanded in a power series in  $(k - k_0)$ , convergent in the metric of  $L_2(\Omega)$ . It turns out that the coefficients of this series are elements of  $C(\Omega)$ , and the series itself, for sufficiently small  $(k - k_0)$ , converges uniformly with respect to  $\vec{\omega} \in \Omega$ . This, in particular, means that for  $k_1 \in (-1, 1)$ , uniformly with respect to  $\vec{\omega} \in \Omega$ ,

$$\lim_{k=k_1} \Phi_k(\vec{\omega}) = \Phi_{k_1}(\vec{\omega}).$$

Using the special form of the kernel of the operator  $\hat{g}$  and the analyticity of  $M(k)$  and  $\Phi_k$ , one can obtain the following results:

**Theorem 4.** If  $k \in (-1, 1)$ , then  $M(-k) = M(k)$ ;  $\Phi_k(-\vec{\omega}) = \Phi_{-k}(\vec{\omega})$  for  $\vec{\omega} \in \Omega$ ;  $M(0) = g_0$ ;  $\Phi_0(\vec{\omega}) \equiv 1$  on  $\Omega$ ;  $dM(k)/dk > 0$  for  $k \in (0, 1)$ ;  $dM(k)/dk|_{k=0} = 0$ ;  $\lim_{k \rightarrow 1-0} M(k) = \infty$ . There exists exactly one  $\lambda_0 \in [0, 1)$  such that  $M(\lambda_0) = 1$ . If  $g_0 = 1$ , then  $\lambda_0 = 0$ ; if  $g_0 < 1$ , then  $\lambda_0 > 0$ .

**Theorem 5.** Let  $k \in (-1, 1)$ ,  $x \in L_2(\Omega) \setminus \{\vartheta\}$ ,

$$(1 + k\vec{\omega}\mathbf{n})x(\vec{\omega}) = \hat{g}x(\vec{\omega}).$$

- I. If  $x(\vec{\omega}) \geq 0$  on  $\Omega$ , then either  $k = \lambda_0$  and  $x = a_1\Phi_{\lambda_0}$ , or  $k = -\lambda_0$  and  $x = a_2\Phi_{-\lambda_0}$ , ( $a_{1,2} = \text{const} > 0$ ).
- II. If  $|k| \neq \lambda_0$  and  $x(\vec{\omega})$  is a real function, then  $|k| > \lambda_0$  and  $x(\vec{\omega})$  assumes on  $\Omega$  both positive and negative values.

Denote by  $\mathfrak{N}$  the set of those  $k \in Z_0$  for each of which (1) admits a nontrivial solution  $\Phi \in L_2(\Omega)$ . Then  $\mathfrak{N} = \mathfrak{N}_0 \cup (-\mathfrak{N}_0)$ ,  $\mathfrak{N}_0 \subset [\lambda_0, 1)$ ,  $\lambda_0 \in \mathfrak{N}_0$ , and  $\mathfrak{N}_0$  is at most countable. If  $\mathfrak{N}_0$  is infinite, then 1 is the only limit point of  $\mathfrak{N}_0$ . For  $k \in Z_0$  define the linear bounded operator  $\widehat{W}(k)$ , acting in  $L_2(\Omega)$ , by the rule:

$$\widehat{W}(k)x = (1 + k\vec{\omega}\mathbf{n})x - \hat{g}x, \quad x \in L_2(\Omega).$$

If  $k \in Z_0 \setminus \mathfrak{N}$ , then  $\widehat{W}(k)$  implements a one-to-one mapping of  $L_2(\Omega)$  onto itself. For  $k \in \mathfrak{N}$  put

$$H_k = \{x \mid x \in L_2(\Omega), \widehat{W}(k)x = 0\}.$$

Then  $H_k$  is a finite-dimensional subspace of  $L_2(\Omega)$ . Let  $p_k$  be the dimension of  $H_k$ . Then  $p_{\pm\lambda_0} = 1$ , and for all  $k \in \mathfrak{N}$ ,  $p_k = p_{-k} \geq 1$ . In the subspaces  $H_k$ , with  $k \in \mathfrak{N}$ ,  $k \neq 0$ , bases  $\{\psi_{kp} \mid p = 1, 2, \dots, p_k\}$  may be chosen in such a way that: a)  $\psi_{kp}(\vec{\omega})$  is a real continuous function of  $\vec{\omega} \in \Omega$ ; b)  $(\vec{\omega}\mathbf{n}, \psi_{kp}^2) = -\text{sgn } k$  for all  $p = 1, 2, \dots, p_k$ , and  $((\vec{\omega}\mathbf{n})\psi_{kp}, \psi_{kp'}) = 0$  for  $1 \leq p < p' \leq p_k$ ; c) for all  $k \in \mathfrak{N}$ ,  $k \neq 0$ , and  $p = 1, 2, \dots, p_k$ ,

$$\psi_{(-k)p}(\vec{\omega}) = \psi_{kp}(-\vec{\omega});$$

- d) if  $g_0 < 1$ , then

$$\psi_{\lambda_0 1} = \left| (\vec{\omega}\mathbf{n}, \Phi_{\lambda_0}^2) \right|^{-1/2} \Phi_{\lambda_0}.$$

Let us now turn to the inhomogeneous equation corresponding to (1).

**Theorem 6.** *Let  $G$  be an open set,  $G \subset Z_0$ , and let  $a(k)$  be a function with values in  $L_2(\Omega)$ , holomorphic in  $G$ . For each  $k \in G \setminus \mathfrak{N}$  there exists exactly one solution  $\psi = \psi(k) \in L_2(\Omega)$  of the equation*

$$(1 + k\vec{\omega}\mathbf{n})\psi(k) = \hat{g}\psi(k) + a(k).$$

*This solution is  $\psi(k) = W^{-1}(k)a(k)$ . It is a single-valued analytic function of  $k$  in  $G \setminus \mathfrak{N}$ . Let  $k_0 \in G \cap \mathfrak{N}$ . Then there exists  $r > 0$  such that*

$$S(k_0, r) \equiv \{k \mid |k - k_0| < r\} \subset G, \quad S(k_0, r) \cap \mathfrak{N} = \{k_0\}.$$

*In  $S(k_0, r) \setminus \{k_0\}$  the solution  $\psi(k)$  can be represented as follows:*

a) *if  $g_0 = 1$  and  $k_0 = 0$ , then*

$$\psi(k, \vec{\omega}) = -\frac{3}{4\pi} \frac{1 - g_1}{k^2} (a(0), 1) + \frac{3}{4\pi k} [(a(0), 1)\vec{\omega}\mathbf{n} - (1 - g_1)(a'(0), 1) + (a(0), \vec{\omega}\mathbf{n})] + u(k, \vec{\omega});$$

b) *if  $k_0 \neq 0$ , then*

$$\psi(k, \vec{\omega}) = -\frac{\operatorname{sgn} k}{k - k_0} \sum_{p=1}^{p_{k_0}} (a(k_0), \psi_{k_0 p}) \psi_{k_0 p}(\vec{\omega}) + u(k, \vec{\omega}).$$

*In both cases  $u(k)$  is a function of  $k$  with values in  $L_2(\Omega)$ , holomorphic in  $S(k_0, r)$ .*

Let us return to the operator  $\widehat{U}(k)$ . It turns out that separation of the polar and azimuthal coordinates of the vector  $\vec{\omega}$  occurs in all eigenfunctions of  $\widehat{U}(k)$ . In order to describe this phenomenon accurately, we introduce the following notation. Let

$$\Delta = [-1, 1] \times [0, 2\pi]$$

be a rectangle in the  $(\mu, \varphi)$ -plane, and let  $\sigma$  be an arbitrary orthonormal basis of three-dimensional space with third vector  $\mathbf{n}$ :

$$\sigma = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{n}\}.$$

Let

$$\vec{\psi}_\sigma(\mu, \varphi) = \sqrt{1 - \mu^2} \cos \varphi \mathbf{e}_1 + \sqrt{1 - \mu^2} \sin \varphi \mathbf{e}_2 + \mu \mathbf{n}, \quad (\mu, \varphi) \in \Delta,$$

and let  $f \in C(\Omega)$ . Fix  $\vec{\omega} \in \Omega$  and  $(\mu, \varphi) \in \vec{\psi}_\sigma^{-1}(\vec{\omega})$ . For each integer  $s$ , put

$$\widehat{P}^{(s)} f(\vec{\omega}) = \frac{1}{2\pi} e^{is\varphi} \int_0^{2\pi} e^{-is\varphi'} f(\vec{\psi}_\sigma(\mu, \varphi')) d\varphi'.$$

It turns out that  $\widehat{P}^{(s)}(C(\Omega)) \subset C(\Omega)$ , and moreover the  $\widehat{P}^{(s)}$  can be extended by continuity to all of  $L_2(\Omega)$ . As an operator acting in  $L_2(\Omega)$ , each  $\widehat{P}^{(s)}$  is a

projection operator.  $\widehat{P}^{(s)}\widehat{P}^{(s')} = \delta_{ss'}\widehat{P}^{(s)}$  for all integers  $s$  and  $s'$ . Finally, each  $\widehat{P}^{(s)}$  commutes with  $\widehat{g}$ .

We now note that the function of three variables  $(\mu, \mu', \varphi')$

$$g(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\varphi')$$

is measurable and square-summable on the parallelepiped  $[-1, 1] \times [-1, 1] \times [0, 2\pi]$ . This makes it possible to define, for integral  $s$ , functions  $g_s(\mu, \mu')$  from  $L_2([-1, 1] \times [-1, 1])$  by the formula

$$g_s(\mu, \mu') = \int_0^{2\pi} g(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos\varphi') e^{is\varphi'} d\varphi'.$$

Let us associate with each  $g_s$  the operator  $\widehat{g}_s$  acting in  $L_2(-1, 1)$ :

$$\widehat{g}_s f(\mu) = \int_{-1}^1 g_s(\mu, \mu') f(\mu') d\mu' \quad \text{for } f \in L_2(-1, 1).$$

**Theorem 7.** Let  $k \in Z_0$  and  $\lambda \in \sigma(\widehat{U}(k))$ ,  $\lambda \neq 0$ . If  $x \in L_2(\Omega)$ ,  $\widehat{U}(k)x = \lambda x$ , then among the functions  $x_s = \widehat{P}^{(s)}x$  there is only a finite number of functions  $x_{s_\nu}$ ,  $\nu = 1, 2, \dots, \nu_0$ , distinct from zero. Moreover,

$$\widehat{U}(k)x_{s_\nu} = \lambda x_{s_\nu} \quad \text{and} \quad x = \sum_{\nu=1}^{\nu_0} x_{s_\nu}.$$

There exist functions  $a_\nu \in L_2(-1, 1)$  such that for all  $\vec{\omega} \in \Omega$

$$x_{s_\nu}(\vec{\omega}) = a_\nu(\mu) e^{is_\nu \varphi}, \quad (\mu, \varphi) \in \vec{\psi}_\sigma^{-1}(\vec{\omega}).$$

The functions  $a_\nu(\mu)$  satisfy the equations

$$\lambda(1 + k\mu)a_\nu(\mu) = \widehat{g}_{s_\nu} a_\nu(\mu), \quad \nu = 1, 2, \dots, \nu_0,$$

and depend continuously on  $\mu \in [-1, 1]$ .

The converse is also true: if  $a_\nu$  satisfy the equations from Theorem 7, then any linear combination of the corresponding  $x_{s_\nu}$  is an eigenfunction of  $\widehat{U}(k)$  belonging to  $\lambda$ .

**Theorem 8.** Let  $k \in (-1, 1)$ . For all integral  $s$ ,

$$\widehat{P}^{(s)}\Phi_k = \delta_{s0}\Phi_k.$$

There exists exactly one function  $\varphi_k(\mu)$  of the variable  $\mu \in [-1, 1]$ ,  $\varphi_k \in L_2(-1, 1)$ , such that, for  $\vec{\omega} \in \Omega$ ,

$$\Phi_k(\vec{\omega}) = \varphi_k(\vec{\omega}\mathbf{n}).$$

Moreover,

$$\varphi_k \in C(-1, 1), \quad M(k)(1 + k\mu)x(\mu) = \hat{g}_0x(\mu), \quad \int_{-1}^1 \varphi_k(\mu) d\mu = 2.$$

The equations

$$M(k)(1 + k\mu)\varphi_k(\mu) = \hat{g}_s\varphi_k(\mu)$$

for  $s \neq 0$  admit only the trivial solution  $x(\mu) \equiv 0$ .

The connection of  $\varphi_k$  with  $\Phi_k$  makes it possible to elucidate many properties of  $\varphi_k$ . Deeper results can be obtained by a direct study of the equation for  $\varphi_k$  <sup>(3-5)</sup>.

Received  
14 IX 1965

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