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Abstract

Full Text

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MATHEMATICS

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ON THE BEST QUADRATURE FORMULAS FOR CERTAIN CLASSES OF FUNCTIONS

(Presented by Academician A. N. Kolmogorov, June 7, 1965)

In the present note we shall study problems whose general formulation is given in the works of S. M. Nikol'skii^(1,2).

1. Denote by $W_{L_q}^{(r)}(M; 0, 1)$, where r is a natural number, $1 \leq q \leq \infty$, the class of functions having on the interval $[0, 1]$ an absolutely continuous derivative of order $r - 1$ and, for $1 \leq q < \infty$, a derivative $f^{(r)}(x)$ of order r satisfying the inequality

$$\left(\int_0^1 |f^{(r)}(x)|^q dx \right)^{1/q} \leq M,$$

and for $q = \infty$ a piecewise continuous derivative $f^{(r)}(x)$ satisfying on the given interval the inequality

$$|f^{(r)}(x)| \leq M.$$

S. M. Nikol'skii^(1,2) introduced into consideration quadrature formulas

$$\int_0^1 f dx \simeq \frac{1}{r!} \sum_{k=0}^{m-1} \sum_{l=0}^{r-2} \lambda_k^{(l)} (r-l-1)! f^{(l)}(x_k) = L(f), \quad (1)$$

defined by given nodes x_k ($0 \leq x_0 < x_1 < \dots < x_{m-1} \leq 1$) and coefficients $\lambda_k^{(l)}$.

Suppose that the quadrature formula (1) is exact for every polynomial of degree $r - 1$. For such a quadrature formula it is not difficult to find (see⁽²⁾) the exact estimate

$$\sup_{f \in W_{L_q}^{(r)}(M; 0, 1)} \left| \int_0^1 f dx - L(f) \right| = M \|F_r(t)\|_{L_p} \quad \left(1 \leq q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1 \right), \quad (2)$$

where

$$F_r(t) = \frac{1}{r!} \left[(1-t)^r - \sum_{k=0}^{m-1} \sum_{l=0}^{r-2} \lambda_k^{(l)} K_{r-l}(x_k - t) \right], \quad (3)$$

$$K_s(u) = \begin{cases} u^{s-1}, & \text{for } u \geq 0, \\ 0, & \text{for } u < 0, \end{cases}$$

$$\|F_r(t)\|_{L_p} = \begin{cases} \left(\int_0^1 |F_r(t)|^p dt \right)^{1/p}, & \text{for } 1 \leq p < \infty, \\ \max_{0 \leq t \leq 1} |F_r(t)|, & \text{for } p = \infty. \end{cases}$$

We shall seek, among all possible quadrature formulas (1) exact for every polynomial of degree $r-1$, the best one for the class $W_{L_q}^{(r)}(M; 0, 1)$, i.e. one for which the quantity (2) has the least

value. We shall solve the problem posed under the assumption that r is an even number.

Let us note that the quadrature formula (1) is exact for any polynomial of degree $r-1$ if and only if, for all t , the condition

$$L[(x-t)^{r-1}] = \int_0^1 (x-t)^{r-1} dx,$$

i.e.

$$\sum_{k=0}^{m-1} \sum_{l=0}^{r-2} \lambda_k^{(l)} (x_k - t)^{r-l-1} = (1-t)^r - t^r. \quad (4)$$

is satisfied.

Taking this into account, we obtain that the solution of our problem consists in finding, for fixed m and r , the least value of the quantity (2), if the numbers x_k are varied ($0 \leq x_0 < x_1 < \dots < x_{m-1} \leq 1$) and the $\lambda_k^{(l)}$, which satisfy condition (4). The solution of this problem is based on the use of properties of

the polynomial $R_{r,p}(x)$ of degree r , with leading coefficient equal to one, which deviates least from zero on the interval $[-1, 1]$ in the metric L_p .

Theorem 1. Among all possible quadrature formulas (1), exact for any polynomial of degree $r-1$ (m, r fixed natural numbers, with r an even number), the best for the class $W_{L_q}^{(r)}(M; 0, 1)$ ($1 \leq q \leq \infty$) is the unique formula

$$\int_0^1 f dx \approx L_*(f)$$

with nodes x_k^* and coefficients $\lambda_{k*}^{(l)}$, expressed by the equalities

$$x_k^* = \left(2k + \sqrt[r]{R_{r,p}(1)}\right) h_* \quad (k = 0, 1, \dots, m-1),$$

$$\lambda_{0*}^{(l)} = \frac{h_*^{l+1}}{(r-l-1)!} \left\{ (-1)^{r-l} \frac{r!}{(l+1)!} [R_{r,p}(1)]^{(l+1)/r} + R_{r,p}^{(r-l-1)}(1) \right\}$$

$$(l = 0, 1, \dots, r-2),$$

$$\lambda_{k*}^{(2i+1)} = 0 \quad (i = 0, 1, \dots, (r-4)/2; k = 1, \dots, m-2),$$

$$\lambda_{k*}^{(2i)} = \frac{2h_*^{2i+1}}{(r-2i-1)!} R_{r,p}^{(r-2i-1)}(1) \quad (i = 0, 1, \dots, (r-2)/2; k = 1, \dots, m-2),$$

$$\lambda_{m-1,*}^{(l)} = \frac{h_*^{l+1}}{(r-l-1)!} \left\{ \frac{r!}{(l+1)!} [R_{r,p}(1)]^{(l+1)/r} - R_{r,p}^{(r-l-1)}(-1) \right\}$$

$$(l = 0, 1, \dots, r-2),$$

where

$$h_* = \frac{1}{2 \left[m-1 + \sqrt[r]{R_{r,p}(1)} \right]}.$$

For this formula the sharp estimate holds

$$\sup_{f \in W_{L_q}^{(r)}(M; 0, 1)} \left| \int_0^1 f dx - L_*(f) \right| = M \frac{h_*^r R_{r,p}(1)}{r! \sqrt[r]{rp+1}}.$$

2. Denote by $W_{0L_q}^{(r)}(M; 0, 1)$ the class of functions consisting of all functions of the class $W_{L_q}^{(r)}(M; 0, 1)$ satisfying the additional condition $f(0) = f'(0) = \dots = f^{(r-1)}(0) = 0$. For any quadrature formula (1), regardless of whether it is exact or inexact for all

polynomials of degree $r-1$, it is not difficult to find (see (2)) the exact estimate

$$\sup_{f \in W_{0L_q}^{(r)}(M; 0, 1)} \left| \int_0^1 f dx - L(f) \right| = M \|F_r(t)\|_{L_p} \quad \left(1 \leq q \leq \infty, \frac{1}{p} + \frac{1}{q} = 1 \right), \quad (5)$$

where $F_r(t)$ is still defined by equality (3).

We shall seek, among all possible quadrature formulas (1), the best one for the class $W_{0L_q}^{(r)}(M; 0, 1)$. Our problem, obviously, consists in finding the smallest value of the quantity (5), if one varies the numbers x_k ($0 \leq x_0 < x_1 < \dots < x_{m-1} \leq 1$) and $\lambda_k^{(l)}$, where m and r are fixed natural numbers. We also solve this problem under the assumption that r is an even number.

Theorem 2. *Among all possible quadrature formulas (1), where m, r are fixed natural numbers, with r an even number, the best for the class $W_{0L_q}^{(r)}(M; 0, 1)$ ($1 \leq q \leq \infty$) is the unique formula*

$$\int_0^1 f dx \simeq L_*(f)$$

with nodes x_{k*} and coefficients $\lambda_{k*}^{(l)}$, expressed by means of the equalities

$$x_{k*} = 2(k+1)h_* \quad (k = 0, 1, \dots, m-1),$$

$$\lambda_{k*}^{(2i+1)} = 0 \quad (i = 0, 1, \dots, (r-4)/2; k = 0, 1, \dots, m-2),$$

$$\lambda_{k*}^{(2i)} = \frac{2h_*^{2i+1}}{(r-2i-1)!} R_{r,p}^{(r-2i-1)}(1)$$

$$(i = 0, 1, \dots, (r-2)/2; k = 0, 1, \dots, m-2),$$

$$\lambda_{m-1,*}^{(l)} = \frac{h_*^{l+1}}{(r-l-1)!} \left\{ \frac{r!}{(l+1)!} [R_{r,p}(1)]^{(l+1)/r} - R_{r,p}^{(r-l-1)}(-1) \right\}$$

$$(l = 0, 1, \dots, r - 2),$$

where

$$h_* = \frac{1}{2m + \sqrt[r]{R_{r,p}(1)}}.$$

For this formula the following exact estimate holds:

$$\sup_{f \in W_0^{(r)} L_q(M; 0, 1)} \left| \int_0^1 f dx - L_*(f) \right| = M \frac{h_*^r R_{r,p}(1)}{r! \sqrt[r]{rp + 1}}.$$

Remark. In the case $r = 2$, $1 \leq q \leq \infty$, Theorem 1 was proved by T. A. Shaidaeva ⁽³⁾. For arbitrary even r , $q = 1$, $q = 2$, and $q = \infty$, Theorem 1 was proved by I. I. Ibragimov and R. M. Aliev; they also proved Theorem 2 for arbitrary even r , $q = 2$ (this was reported by R. M. Aliev at the seminar on constructive function theory at the V. I. Lenin Belarusian State University). We note that in the case $r = 2$, $q = 2$, Theorem 2 had earlier also been proved by G. Ya. Doronin ⁽⁴⁾. For arbitrary even r and $q = \infty$, Theorem 2 was proved by S. M. Nikol'skii ^(1,2), and for $q = 1$ by A. I. Kiselev (see ⁽²⁾, pp. 107-109).

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