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Abstract

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MATHEMATICS

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ON SOLUTIONS IN GENERALIZED FUNCTIONS OF ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

(Presented by Academician I. G. Petrovskii, 11 XI 1965)

In the paper ⁽¹⁾ it was proved that an equation of the form

$$L(y) \equiv \sum_{k=0}^m a_k x^{k+r} y^{(k)}(x) + \sum_{q=0}^n b_q x^q y^{(q)}(x) = 0, \quad m > n, \quad r > 0,$$

in the space of generalized functions $(S_0^\beta)'$, with some $\beta > 1$, has a fundamental system of $2m + r$ linearly independent solutions. In the present note we study the more general equation

$$L(y) \equiv \sum_{p=0}^n \sum_{k=0}^{m_p} a_{kp} x^{k+p} y^{(k)}(x) = 0, \quad (1)$$

$$m_n > m_p > 0, \quad p = 0, 1, \dots, n-1, \quad a_{m_n n} \neq 0, \quad a_{00} \neq 0,$$

in the same space $(S_0^\beta)'$, $\beta > 1$.

Denote by Ω_0 the totality of all solutions of equation (1) in $(S_0^\beta)'$ concentrated at the point $x = 0$.

Theorem 1. *The dimension of the subspace $\Omega_0 \subset (S_0^\beta)'$ is equal to n .*

Proof. Let

$$y(x) = \sum_{q=0}^{\infty} c_q \delta^{(q)}(x). \quad (2)$$

Under the condition of convergence of the series (2), this expression is the general form of a functional in $(S_0^\beta)'$ concentrated at the point $x = 0$ (see (2)). To prove the theorem it is enough to determine how many free constants c_q there are in the expansion (2) such that expression (2), with these free constants, is a solution of equation (1). Substituting expression (2) into equation (1) and using the formulas $x^k \delta^{(q)}(x) = (-1)^k [q]_k \delta^{(q-k)}(x)$ for $k < q$ ($= 0$ for $k > q$), by virtue of the linear independence of $\delta^{(q)}(x)$ for different q , equating to zero the resulting coefficient of $\delta^{(q)}(x)$, we obtain

$$\sum_{p=0}^n D_p(q) c_{q+p} = 0, \quad q = 0, 1, \dots, \quad (3)$$

where

$$D_p(q) = \sum_{k=0}^{m_p} a_{kp} (-1)^{k+p} [q+k+p]_{k+p}, \quad [q]_\rho = q(q-1) \cdots (q-\rho+1).$$

We denote the degree of the polynomial $D_p(q)$ by d_p . Obviously, $d_p = m_p + p$, and, by virtue of our assumptions, $d_n > d_p$, $p = 0, 1, \dots, n-1$. It is clear that in equation (3), assigning $c_0, c_1, \dots, c_{n-1}$ arbitrarily, it is easy to determine subsequent c_q such that

$$c_{q+n} = - \sum_{p=0}^{n-1} \frac{D_p(q)}{D_n(q)} c_{q+p}, \quad q = 0, 1, 2, \dots \quad (4)$$

It remains for us to verify that the series (2) constructed from the coefficients (4) thus obtained converges in $(S_0^\beta)'$ with some $\beta > 1$. For this it is sufficient that, as $q \rightarrow \infty$, the coefficients c_q satisfy the estimates

$$|c_q| \leq \frac{B^q}{q^{qr_0}} M_q, \quad r_0 > 1, \quad B > 1, \quad M_q < c. \quad (5)$$

Indeed, under such an estimate for c_q we have

$$\begin{aligned} \left| \sum_{q=0}^{\infty} (c_q \delta^{(q)}(x), \varphi(x)) \right| &= \left| \sum_{q=0}^{\infty} c_q (-1)^q \varphi^{(q)}(0) \right| \leq \sum_{q=0}^{\infty} |c_q| |\varphi^{(q)}(0)| \leq \\ &\leq \sum_{q=0}^{\infty} \frac{B^q B_1^q q^{q\beta}}{q^{qr_0}} M_q \leq \sum_{q=0}^{\infty} \left(\frac{\bar{B}}{q^{r_0-\beta}} \right)^q < \infty \quad \text{for } \beta < r_0. \end{aligned}$$

Let us prove the estimates (5). Denoting $M_q = c_q q^{qr_0} / B^q$, from (4) we obtain

$$M_{q+n} = -\frac{1}{B^{q+n}} \sum_{p=0}^{n-1} \frac{D_p(q)}{D_n(q)} \frac{B^q(q+n)^{(q+n)r_0}}{(q+p)^{(q+p)r_0}} M_{q+p}.$$

For

$$r_0 = \min_{0 < p < n-1} \left\{ 1 + \frac{m_n - m_p}{n - p} \right\}$$

there exists such a q_1 that for $q > q_1$

$$M_{q+n} \leq \frac{c}{(1+q)^{\varepsilon+1}B} \sum_{p=0}^{n-1} M_{q+p}.$$

Let $\bar{M}_{n-1} = \max(M_0, M_1, \dots, M_{n-1})$; then

$$M_{q+n} \leq \frac{c_n}{(1+q)^{\varepsilon}} \frac{1}{B} \bar{M}_{q+n-1};$$

choosing $B > cn$, we obtain $M_{q+n} \leq \bar{M}_{q+n} \leq \bar{M}_{q+n-1}$, which means boundedness of M_q as $q \rightarrow \infty$. This proves Theorem 1 completely.

Put

$$f^\lambda(x) = |x|^\lambda / \Gamma\left(\frac{\lambda+1}{2}\right), \quad g^\lambda(x) = |x|^\lambda \operatorname{sign} x / \Gamma\left(\frac{\lambda+2}{2}\right).$$

As is known ([3], p. 81 and following), by analytic continuation these functions define a functional on $S' \supset (S_0^\beta)'$ for all values of λ ; in particular, for $\lambda = -(2n+1)$ and $\lambda = -2n$ we obtain

$$f^{-(2n+1)}(x) = (-1)^n n! \delta^{(2n)}(x) / (2n)!, \quad g^{-2n}(x) = \frac{(-1)^n (n-1)! \delta^{(2n-1)}(x)}{(2n-1)!}.$$

The following formulas for differentiating and multiplying by powers of x the functionals $f^\lambda(x)$ and $g^\lambda(x)$ hold:

$$x^q [f^\lambda(x)]^{(r)} = \begin{cases} 2^r \left[\frac{\lambda}{2} \right]_{E(r/2+1)} \left[\frac{\lambda+q-r-1}{2} \right]_{E(q/2)} f^{\lambda+q-r}(x), & \text{if } q+r \text{ is even,} \\ 2^r \left[\frac{\lambda}{2} \right]_{E(r/2+1)} \left[\frac{\lambda+q-r}{2} \right]_{E(q/2+1)} g^{\lambda+q-r}(x), & \text{if } q+r \text{ is odd;} \end{cases} \quad (6)$$

$$x^q[g^\lambda(x)]^{(r)} = \begin{cases} 2^r \left[\frac{\lambda-1}{2} \right]_{E(r/2)} \left[\frac{\lambda+q-r}{2} \right]_{E(q/2+1)} g^{\lambda+q-r}(x), & \text{if } q+r \text{ is even,} \\ 2^r \left[\frac{\lambda-1}{2} \right]_{E(r/2)} \left[\frac{\lambda+q-r-1}{2} \right]_{E(q/2)} f^{\lambda+q-r}(x), & \text{if } q+r \text{ is odd.} \end{cases} \quad (7)$$

Here $E(\lambda)$ is the integer part of λ .

Put

$$R_p(\lambda) = \sum_{k=0}^{m_p} a_{kp}[\lambda-p]_k, \quad A_p(\lambda) = \left[\frac{\lambda+1}{2} \right]_{E(p/2)} R_p(\lambda),$$

$$B_p(\lambda) = \left[\frac{\lambda}{2} \right]_{E(p/2+1)} R_p(\lambda).$$

We shall call the polynomial $A_n(\lambda)$ the characteristic polynomial of equation (1).

Definition. We shall say that the polynomial $A(\lambda)$ has no generalized multiple roots modulo r , if the arithmetic progressions with difference r constructed on the roots of the polynomial $A(\lambda)$ do not intersect. If, however, the arithmetic progression with difference r constructed over the root λ_1 contains k roots of the polynomial $A(\lambda)$, then we shall say that λ_1 is a generalized multiple root modulo r of the polynomial $A(\lambda)$ of order k .

Theorem 2. If the polynomial $A_n(\lambda)$ has no generalized multiple roots modulo n for even n and modulo $n-1$ for odd n , then equation (1) in the space $(S_0^\beta)'$ with some $\beta > 1$ has $2m_n + n$ linearly independent solutions of the form

$$y(x) = \sum_{\lambda} \xi_{\lambda} f^{\lambda}(x) + \sum_{\lambda} \eta_{\lambda} g^{\lambda}(x), \quad (8)$$

where the summation index λ runs through the values from a certain arithmetic progression with difference n for even n and with difference $n-1$ for odd n .

Proof. Substituting the expressions (8) into equation (1) and using formulas (6), (7), in view of the linear independence of $f^{\lambda}(x)$ and $g^{\lambda}(x)$ for different λ , we obtain

$$\sum_{p=0}^n {}''A_p(\lambda) \xi_{\lambda-p} + \sum_{p=0}^n {}'A_p(\lambda) \eta_{\lambda-p} = 0, \quad (9)$$

$$\sum_{p=0}^n {}'' B_p(\lambda) \eta_{\lambda-p} + \sum_{p=0}^n {}' B_p(\lambda) \xi_{\lambda-p} = 0. \quad (10)$$

Here two primes and one prime over $\sum$ mean that the summation extends over even and odd p , respectively. The pair of equations (3), (9) is equivalent to equation (1). Suppose that n is even. In this case the degrees a_n and b_n of the polynomials $A_n(\lambda)$ and $B_n(\lambda)$ are equal to each other, namely $a_n = b_n = (n/2 + m_n)$.

Using the structure of the polynomials $A_p(\lambda)$ and $B_p(\lambda)$ and substituting in (9), (10) the values $\lambda = 1, 3, 5, \dots, 0, 2, 4, \dots$, it is not difficult to see that $\xi_{2k+1} = \eta_{2k} = 0$ for the values $k = 0, 1, 2, \dots$. Further, assigning $\xi_{-1}, \xi_{-3}, \dots, \xi_{-(n-1)}$ and $\eta_{-2}, \eta_{-4}, \dots, \eta_{-(n-2)}$ arbitrarily, it is easy to determine all values ξ_k lying on arithmetic progressions with difference n constructed over the numbers $\xi_{-1}, \xi_{-3}, \dots, \xi_{-(n-1)}$, and all values η_k lying on arithmetic progressions with difference n constructed over the numbers $\eta_{-2}, \eta_{-4}, \dots, \eta_{-(n-2)}$. The ξ_k and η_k found in this way depend only on the values $\xi_{-1}, \xi_{-3}, \xi_{-5}, \dots, \xi_{-(n-1)}$; $\eta_{-2}, \eta_{-4}, \dots, \eta_{-(n-2)}$ and determine n linearly independent solutions of equation (1). These solutions are concentrated at the point $x = 0$.

Let now λ be a root of the polynomial $R_n(\lambda)$. Assigning $\xi_{\lambda_1-n}, \eta_{\lambda_1-n}$ arbitrarily and $\xi_{\lambda_1-p-kn} = \eta_{\lambda_1-p-kn} = 0$ for the values $p = 0, 1, \dots, n-1$, $k = 0, 1, 2, \dots$, we find all values ξ_λ and η_λ belonging to arithmetic progression constructed above λ_1 with difference n . In this case we obtain

$$\xi_{\lambda_1-kn} \leq \frac{A_0(\lambda_1 - (k-1)n)}{A_n(\lambda_1 - (k-1)n)} \xi_{\lambda_1 - (k-1)n},$$

whence

$$|\xi_{\lambda_1-kn}| \leq \frac{c}{k^{k(m_n+n+m_0)}} \xi_{\lambda_1-n}. \quad (11)$$

Similarly,

$$|\eta_{\lambda_1-kn}| \leq \frac{c}{k^{k(m_n+n-m_0)}} \eta_{\lambda_1-n}. \quad (12)$$

The values ξ_λ and η_λ thus found determine, by formula (8), two solutions of equation (1). Since $A_n(\lambda)$ has no multiple generalized roots, these solutions are linearly independent. The convergence of the series (8) with these ξ_λ and η_λ follows from the estimates (11), (12). Carrying out the indicated procedure for all roots of the polynomial $R_n(\lambda)$, we obtain another $2m_n$ linearly independent solutions of equation (1).

Thus, for even n , equation (1) has $2m_n + n$ linearly independent solutions of the form (8), of which n are concentrated at the point $x = 0$.

For odd n the proof is completely analogous, except that in this case $a_n = (n - 1)/2 + m_n$, $b_n = (n + 1)/2 + m_n$, and the arithmetic progressions will be constructed not with difference n , but with difference $n - 1$. This completes the proof of Theorem 2.

Theorem 3. *The linearly independent solutions found in Theorem 2 form a fundamental system of solutions of equation (1) in $(S_0^\beta)'$, $\beta > 1$.*

The proof of Theorem 3 is carried out in the same way as the proof of the analogous assertion (Theorems 2 and 3) in paper ⁽¹⁾.

Example. The equation

$$x^3 y' + ({}^{3/2}_2 x^2 + x + 1) y = 0$$

has in $(S_0^{3/2})'$ the fundamental system of solutions

$$y_1(x) = \delta(x) + \frac{1}{5} \delta''(x) - \frac{4}{7 \cdot 5 \cdot 3!} \delta^{IV}(x) + \dots$$

$$y_2(x) = -\delta' + \frac{1}{5} \delta'''(x) - \frac{2}{5 \cdot 3!} \delta^V(x) + \dots$$

$$y_3(x) = \sum_{k=0}^{\infty} \frac{2^k (-1)^k}{k! \prod_{\mu=1}^k (4\mu - 1)} f^{-(1+4k)/2}(x),$$

$$y_4(x) = \sum_{n=0}^{\infty} \frac{(-1)^k 2k}{k! \prod_{\mu=0}^n (4\mu + 1)} g^{-(1+4k)/2}(x).$$

The solutions y_1 and y_2 are concentrated at the point $x = 0$.

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References

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Note: Figure translations are in progress. See original paper for figures.

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