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N. N. Krasovskii

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Abstract

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MECHANICS

Corresponding Member of the Academy of Sciences of the USSR N. N. KRASOVSKII

ON THE APPROXIMATION OF A PROBLEM OF OPTIMAL CONTROL IN A SYSTEM WITH AFTEREFFECT

Consider the problem ^(1,2) of optimal control $u^0[t, x(t + \vartheta)]$, which stabilizes, up to asymptotic stability, the system with aftereffect

$$\dot{x}(t) = \int_{-\sigma}^0 d_{\vartheta} H(t, \vartheta) x(t + \vartheta) + B(t)u \quad (1)$$

and at the same time minimizes the integral

$$I_{\infty}[t_0, x_0(\vartheta), u] = \int_{t_0}^{\infty} [\|x(t)\|^2 + \|u(t)\|^2] dt. \quad (2)$$

Here $x = \{x_i\}$ is an n -dimensional vector of phase coordinates of the controlled object; $x(t + \vartheta)$ ($-\sigma \leq \vartheta \leq 0$) is an element of the trajectory of system (1) of temporal length $\sigma > 0$ —const; $H(t, \vartheta)$ is a matrix with bounded variation in ϑ for $-\sigma \leq \vartheta \leq 0$; $B(t)$ is a matrix bounded for $t \geq 0$; u is an r -dimensional vector of the control force; $x_0(\vartheta)$ is the initial perturbation $x(t_0 + \vartheta)$ ($-\sigma \leq \vartheta \leq 0$) of the object; the symbols $\|x\|$ and $\|u\|$ denote the Euclidean norms of the vectors x and u .

The optimal control $u^0[t, x(t + \vartheta)]$ for problem (1), (2) has the form of a linear functional ^(2,3)

$$u^0[t, x(t + \vartheta)] = P(t)x(t) + \int_{-\sigma}^0 Q(t, \vartheta)x(t + \vartheta)d\vartheta,$$

the concrete computation of which, however, is associated with difficulties. Therefore there arises the problem of approximating problem (1), (2) by a suitable finite-dimensional problem. One such approximation was proposed in ⁽⁴⁾ and investigated in the case of the proper integral (2) and the stationary

system (1) in ⁽⁵⁾. The results given below generalize the results from article ⁽⁵⁾.

Let certain finite-dimensional systems be associated with system (1)

$$\dot{y}^{(m)}(t) = A^{(m)}(t)y^{(m)}(t) + B^{(m)}(t)v, \quad (3)$$

and let integral (2) be approximated by the integrals

$$I_{\infty}^{(m)}[t_0, y_0^{(m)}, v] = \int_{t_0}^{\infty} [\|\xi^{(m)}(t)\|^2 + \|v(t)\|^2] dt. \quad (4)$$

Here $y^{(m)}$ are k_m -dimensional vectors; $A^{(m)}(t)$ and $B^{(m)}(t)$ are, respectively, $(k_m \times k_m)$ - and $(k_m \times r)$ -matrices; v is an r -vector; $\xi^{(m)}$ is an n -dimensional vector that is a linear function of $y^{(m)}$.

We shall consider the motion of system (1) in the space of functions $x(\vartheta)$ ($-\sigma \leq \vartheta \leq 0$), taking as an element of motion the function $x(t + \vartheta)$ ($-\sigma \leq \vartheta \leq 0$). The right-hand side of (1) is defined for functions continuous in ϑ

functions $x(t + \vartheta)$. However, many equations (1) retain meaning also in spaces broader than the space $C[-\sigma, 0]$ of continuous functions $x(\vartheta)$. We shall therefore assume that some space X of functions $x(\vartheta)$ ($-\sigma \leq \vartheta \leq 0$) has been chosen (possibly broader than $C[-\sigma, 0]$), with norm $\rho[x(\vartheta)]$, where the solutions of equation (1) considered below exist. We shall also assume that in the spaces $Y^{(m)}$ of vectors $y^{(m)}$ certain norms $\rho^{(m)}[y^{(m)}]$ have been chosen. Systems (1) and (3) are connected by a linear function $y^{(m)} = f^{(m)}[x(\vartheta)]$, defined on X . The problem consists in estimating the approximation of problem (1), (2) by problem (3), (4). The central question is the following: for large m , does the control of system (1) according to the law $u = v^{(m)0}[t, f^{(m)}[x(t + \vartheta)]]$ give an effect close to the optimal one given by the control $u = u^0[t, x(t + \vartheta)]$.

We introduce the following notation. Let $x(t, t_0, x_0(\vartheta), u)$ and $y^{(m)}(t, t_0, y_0^{(m)}, v)$ be, respectively, the motions of systems (1) and (3) generated by the initial conditions $x(t_0 + \vartheta) = x_0(\vartheta)$ ($-\sigma \leq \vartheta \leq 0$), $y^{(m)}(t_0) = y_0^{(m)}$, under certain chosen controls $u(t)$ and $v(t)$, or $u[t, x(t + \vartheta)]$, $v[t, y^{(m)}(t)]$. By the symbol $\xi^{(m)}(t, t_0, y_0^{(m)}, v)$ we denote the functions $\xi^{(m)}[y^{(m)}(t, t_0, y_0^{(m)}, v)]$. We denote optimal controls for problems (1), (2) and (3), (4), respectively, by the symbols $u^0[t, x(t + \vartheta)]$ and $v^{(m)0}[t, y^{(m)}(t)]$. The same controls, but considered as functions of time $t \geq t_0$ on the motions $x(t, t_0, x_0(\vartheta), u^0)$ and $y^{(m)}(t, t_0, y_0^{(m)}, v^{(m)0})$, will be denoted by $w^0(t, t_0, x_0(\vartheta))$ and $w^{(m)0}(t, t_0, y_0^{(m)})$. We shall denote the response of system (1) to the impulse control $u_k(t) = \delta(t - t_0)$, $u_j(t) \equiv 0$ for $j \neq k$ ($\delta(t)$ is the delta function), by the symbol $h[t, t_0]_{(k)}$; the response of the motion $\xi^{(m)}(t)$ of system (3) to the action $v_k(t) = \delta(t - t_0)$, $v_j(t) \equiv 0$ for $j \neq k$, will be denoted by $h^{(m)}[t, t_0]_{(k)}$. The vectors under consideration are interpreted as

column vectors. The upper index asterisk denotes transposition. Suppose that s -vector functions $w(t)$ are considered on the interval $[\alpha, \beta]$. We shall denote by the symbols $\mathcal{L}_s^2[\alpha, \beta]$, $\mathcal{L}_s^2[\alpha, \beta]$, and $C_s[\alpha, \beta]$, respectively, the functional spaces with norms

$$\|w(t)\|_{[\alpha, \beta]} = \left[\int_{\alpha}^{\beta} \|w(t)\|^2 dt \right]^{1/2},$$

$$\|w(t)\|_{[\alpha, \beta]} = \left[\|w(\beta)\|^2 + \int_{\alpha}^{\beta} \|w(t)\|^2 dt \right]^{1/2},$$

$$\|w(t)\|_{[\alpha, \beta]}^C = \sup[\|w(t)\| \text{ for } \alpha \leq t \leq \beta].$$

We assume that X coincides with $\mathcal{L}_n^2[-\sigma, 0]$. This choice is convenient for many problems. In the case of choosing another space X , corresponding changes are introduced into the conditions below. Let us formulate the conditions imposed on systems (1) and (3).

Condition 1. For every $\tau > 0$, the functions $h^{(m)}[t, t_0]_{(k)}$ and $\xi^{(m)}(t, t_0, f^{(m)}[x_0(\vartheta)], v \equiv 0)$ converge to the functions $h[t, t_0]_{(k)}$ and $x(t, t_0, x_0(\vartheta), u \equiv 0)$, respectively, in the metric of $\mathcal{L}_n^2[t_0, T)$ for all $t_0 \geq 0$ and $x_0(\vartheta) \in X_0$ (X_0 is a subspace in X with norm $\rho_0[x_0(\vartheta)]$). Moreover, for $\tau \geq \tau_0 = \text{const}$, the convergence of $h^{(m)}(t)$ and $\xi^{(m)}(t)$ to $h(t)$ and $x(t)$ also takes place in $C[t_0 + \tau_0, t_0 + \tau]$. Both convergences are uniform with respect to $t_0 \geq 0$ and $\rho_0[x_0(\vartheta)] \leq 1$.

Condition 2. Let γ be any fixed positive number. If $\rho^{(m)}[y_0^{(m)}] \leq \gamma$ and $\|\xi^{(m)}(t)\|_{[t_0, T]} \leq \gamma$, then for any $\varepsilon > 0$ one can indicate $\delta > 0$ and N such that

$$\rho^{(m)}[y^{(m)}(T, t_0, y_0^{(m)})] < \varepsilon,$$

provided only that

$$\|\xi^{(m)}(t, t_0, y_0^{(m)}, v)\|_{[T-\beta, T]}^C \leq \delta$$

and $m \geq N$. Here $\beta > 0$ is a constant independent of γ , $T \geq t_0 + \beta$.

Condition 3. Systems (1) and (3) are uniformly stabilizable for all sufficiently large values of m , i.e., there exist numbers μ and N satisfying the condition: whatever the initial data $t_0 \geq 0$, $x_0(\vartheta) \in X$ with $\rho[x_0(\vartheta)] \leq 1$, and $y_0^{(m)} \in Y^{(m)}$ with $\rho^{(m)}[y_0^{(m)}] \leq 1$, and $m \geq N$, there exist controls $u_*(t, t_0, x_0)$ and $v_*^{(m)}(t, t_0, y_0^{(m)})$, for which

$$I[t_0, x_0(\vartheta), u_*] \leq \mu \quad \text{and} \quad I^{(m)}[t_0, y_0^{(m)}, v_*^{(m)}] \leq \mu.$$

Suppose conditions 1–3 are fulfilled. Then the following assertions are valid.

Lemma 1. Uniformly with respect to $t_0 \geq 0$ and $\rho_0[x_0(\vartheta)] \leq 1$,

$$\lim_{m \rightarrow \infty} I_{\infty}^{(m)}[t_0, f^{(m)}[x_0(\vartheta)], v^{(m)0}] = I_{\infty}[t_0, x_0(\vartheta), u^0]. \quad (5)$$

A limiting relation of the form (5) is first proved for proper integrals, $\lim I_T^{(m)} = I_T$, on the basis of condition 1. Then, on the basis of conditions 2 and 3, this relation is extended to the improper integrals $I_{\infty}^{(m)}$.

Lemma 2. Uniformly with respect to $t_0 \geq 0$ and $\rho_0[x_0(\vartheta)] \leq 1$,

$$\lim_{m \rightarrow \infty} \|w^{(m)0}(t, t_0, f^{(m)}[x_0(\vartheta)]) - w^0(t, t_0, x_0(\vartheta))\|_{[t_0, \infty)} = 0. \quad (6)$$

The limiting relation (6) is proved from Lemma 1 and from an estimate of the second variation of the functional $I_{\infty}[t_0, x_0(\vartheta), u]$, generated by the variation $\delta u = w^{(m)0}(t) - w^0(t)$.

Theorem 1. Uniformly with respect to $t \geq 0$ and $\rho_0[x(\vartheta)] \leq 1$,

$$\lim_{m \rightarrow \infty} \|v^{(m)0}[t, f^{(m)}[x(\vartheta)]] - u^0[t, x(\vartheta)]\| = 0. \quad (7)$$

The theorem is proved from Lemma 2 and from the integral equations (5), which are satisfied by the optimal controls $w_j^0(t)$ and $w_j^{(m)0}(t)$ ($j = 1, \dots, r$):

$$w_j^0(t) = - \int_t^{\infty} h^*[\tau, t]_{(j)} \left(\int_{t_0}^{\tau} \left[\sum_{k=1}^r h[\tau, \xi]_{(k)} w_k^0(\xi) \right] d\xi + x(\tau, t_0, x_0(\vartheta), u \equiv 0) \right) d\tau, \quad (8)$$

$$w_j^{(m)0}(t) = - \int_t^{\infty} h^{(m)*}[\tau, t]_{(j)} \left(\int_{t_0}^{\tau} \left[\sum_{k=1}^r h^{(m)}[\tau, \xi]_{(k)} w_k^{(m)0}(\xi) \right] d\xi + \xi^{(m)}(\tau, t_0, f^{(m)}[x_0(\vartheta)], v \equiv 0) \right) d\tau. \quad (9)$$

Here it is taken into account that, under conditions 1–3, the convergence of the integrals on the right-hand sides of (8) and (9) is sufficiently uniform for it to be possible, for large m , to approximate them by proper integrals.

If the control $u = u^0[t, x(t + \vartheta)]$ uniformly stabilizes system (1) in X_0 , and the control $u = v^{(m)0}[t, f^{(m)}[x(t + \vartheta)]]$, for $x_0(\vartheta) \in X_0$, does not take the motion $x(t + \vartheta)$ out of X_0 for $t \geq t_0$, and moreover

$$\|\rho_0[x(t, t_0, x_0(\vartheta), u) - x(t, t_0, x_0(\vartheta), v)]\| \leq \|u - v\| \gamma$$

for $t = t_0$, then from Theorem 1 there follow the limiting relations

$$\lim_{m \rightarrow \infty} I[t_0, x_0(\vartheta), v^{(m)0}[t, f^{(m)}[x(t + \vartheta)]]] = I[t_0, x_0(\vartheta), u^0],$$

$$\lim_{m \rightarrow \infty} \|x(t, t_0, x_0(\vartheta), v^{(m)0}) - x(t, t_0, x_0(\vartheta), u^0)\|_{[t_0, \infty]}^c = 0,$$

which hold uniformly with respect to $t_0 \geq 0$ and $\rho_0[x_0(\vartheta)] \leq 1$.

For the approximation considered in article (5), conditions 1–3 are fulfilled, for example, for $X_0 = C[-\sigma, 0]$.

Sverdlovsk Branch
of the V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

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