

ON A PROBLEM OF BIOPHYSICS

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Abstract

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MATHEMATICS

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ON A PROBLEM OF BIOPHYSICS

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The investigation of the question of the distribution of concentrations of substances participating in the vital processes of a living cell leads (see ⁽¹⁾) to the problem for a system of diffusion equations

$$\begin{aligned} a_{11}\Delta u_{11} - (c_2 + c_3)u_{11} - \partial u_{11}/\partial t &= 0, & a_{21}\Delta u_{21} + c_2u_{11} - \partial u_{21}/\partial t &= 0, \\ (x, t) &\in D_T^{(1)}, \\ a_{k2}\Delta u_{k2} - \partial u_{k2}/\partial t &= 0 \quad (k = 1, 2), & (x, t) &\in D_T^{(2)} \end{aligned} \quad (1)$$

with initial conditions for $t = 0$ for $u_{kl}(x, t)$, linear boundary conditions for u_{k2} on $\Gamma^{(2)}$ —the outer boundary of the external medium $D_T^{(2)}$ —and conjugation conditions on the surface $\Gamma^{(1)}$ of the cell membrane $D_T^{(1)}$, immersed in the medium $D_T^{(2)}$:

$$(-1)^l a_{kl} \partial u_{kl}(x, t) / \partial N_l(x, t) - h_k (u_{k2}(x, t) - u_{k1}(x, t)) = 0, \quad k, l = 1, 2, \quad (x, t) \in \Gamma^{(1)}. \quad (2)$$

Here $u_{kl}(x, t)$ in (1) is the concentration of the substance S_k inside ($l = 1$) and outside ($l = 2$) the cell. S_1 is the substance entering the cell $D_T^{(1)}$ from the external medium $D_T^{(2)}$ and supplying the organism with energy. Under enzymatic transformations, part of S_1 passes into low-molecular substances S_2 , which are removed from the cell, while from another part of S_1 the cell synthesizes its specific substances S_3 . The constants $c_i > 0$ characterize the ability of the enzymes of the cell biomass to transform S_1 into S_i for $i = 2, 3$. $a_{kl} > 0$ ($k, l = 1, 2$) is the diffusion coefficient of the substance S_k inside ($l = 1$) and outside ($l = 2$) the cell. The constant $h_k > 0$ in (2) characterizes the permeability of the cell membrane $\Gamma^{(1)}$ for the substance S_k ($k = 1, 2$). $N_l(x, t)$ is the normal, internal with respect to $D_T^{(l)}$, to the section $\Gamma^{(1)} \cap \{t = t > 0\}$.

We shall consider a more general problem for a system of parabolic equations with discontinuous coefficients of a special kind, preserving the specificity of the system (1)–(2):

$$\sum_{i,j=1}^n a_{ij}^{(kl)}(x,t) \frac{\partial^2 u_{kl}}{\partial x_i \partial x_j} + \sum_{i=1}^m \sum_{j=1}^n b_{ij}^{(kl)}(x,t) \frac{\partial u_{il}}{\partial x_j} + \sum_{i=1}^m c_i^{(kl)}(x,t) u_{il} - \partial u_{kl} / \partial t = f_{kl}(x,t), \quad k = 1, 2, \dots, m; \quad l = 1, 2, \quad (3)$$

with supplementary conditions

$$u_{kl}(x, 0) = f_{kl}^{(1)}(x), \quad x \in \Omega^{(l)} = \overline{D_T^{(l)}} \cap \{t = 0\}, \quad k = 1, 2, \dots, m; \quad l = 1, 2; \quad (4)$$

$$a_k^{(k)}(x,t) \frac{\partial u_{k2}(x,t)}{\partial \nu_k(x,t)} - b_k^{(k)}(x,t) u_{k2}(x,t) = \sum_{i \neq k, i=1}^m \left(-a_i^{(k)}(x,t) \frac{\partial u_{i2}(x,t)}{\partial \nu_i(x,t)} + b_i^{(k)}(x,t) u_{i2}(x,t) \right) + f_k^{(2)}(x,t), \quad (x,t) \in \Gamma^{(1)} \quad (5)$$

$$\begin{aligned} & (-1)^l d_{kl}^{(kl)}(x,t) \frac{\partial u_{kl}(x,t)}{\partial \nu_{kl}(x,t)} - h_{kl}^{(kl)}(x,t) (u_{k2}(x,t) - u_{k1}(x,t)) = \\ & = \sum_{i \neq k, i=1}^m \left[(-1)^{l+1} d_{il}^{(k)}(x,t) \frac{\partial u_{il}(x,t)}{\partial \nu_{il}(x,t)} + \sum_{j=1}^2 h_{ij}^{(kl)}(x,t) u_{ij}(x,t) \right] + f_{kl}^{(3)}(x,t), \quad (6) \end{aligned}$$

$$(x,t) \in \Gamma^{(1)}, \quad k = 1, 2, \dots, m; \quad l = 1, 2.$$

Throughout what follows we shall use the notation and definitions of works (2–4). $D_T^{(1)} \cup \Gamma^{(1)} \cup D_T^{(2)} = D_T$ is a bounded domain of $(n+1)$ -dimensional Euclidean space $(x,t) = (x_1, x_2, \dots, x_n, t)$, lying between the two hyperplanes $t = 0$ and $t = T > 0$, having as its lower base the domain $\Omega^{(1)} \cup \Omega^{(2)}$ and as its lateral boundary the closed surface $\Gamma^{(2)}$; moreover $D_T^{(1)}$ is an interior subdomain of D_T with lateral boundary surface $\Gamma^{(1)}$, separating $D_T^{(1)}$ and $D_T^{(2)}$. The closed surfaces $\Gamma^{(l)}$ ($l = 1, 2$), of Lyapunov type, the tangent planes to which are nowhere orthogonal to the axis Ot , are situated between the hyperplanes $t = 0$ and $t = T$ and do not intersect one another, being separated from each other by a distance not less than $d > 0$. On $\Gamma^{(1)}$ there are prescribed fields of directions with unit vectors $\nu_{il}(x,t)$ ($i = 1, 2, \dots, m; \quad l = 1, 2$), lying in the section $\Omega_t^{(l)} = D_T^{(l)} \cap \{\tau = t\}$ and forming acute angles not exceeding $\pi/2 - d_0$ ($d_0 > 0$) with the normal $N_{1l}(x,t)$, interior with respect to $\Omega_t^{(l)}$, at the point

$(x, t) \in \Gamma^{(1)}$. On $\Gamma^{(2)}$ there are prescribed fields of directions with unit vectors $\nu_i(x, t)$ ($i = 1, 2, \dots, m$), lying in the section $\Omega_t^{(2)}$ and forming acute angles not exceeding $\pi/2 - d_0$ with the interior normal $N_2(x, t)$ at the point $(x, t) \in \Gamma^{(2)}$.

In conditions (5), (6) the inequalities are always assumed to hold

$$a_k^{(k)}(x, t) \geq a_0 > 0, \quad b_k^{(k)}(\bar{x}, t) \geq 0, \quad d_{kl}^{(k)}(\bar{x}, t) \geq \delta > 0, \quad h_{kl}^{(kl)}(\bar{x}, t) \geq 0,$$

$$(x, t) \in \Gamma^{(2)}, \quad (\bar{x}, t) \in \Gamma^{(1)}, \quad k = 1, 2, \dots, m; \quad l = 1, 2.$$

§ 1. Suppose that for (3)–(6) the additional conditions are satisfied

$$b_{ij}^{(kl)}(x, t) \equiv c_i^{(kl)}(x, t) \equiv 0 \text{ in (3)}, \quad a_i^{(k)}(x, t) \equiv b_i^{(k)}(x, t) \equiv 0 \text{ in (5)},$$

$$d_{il}^{(k)}(x, t) \equiv h_{ij}^{(kl)}(x, t) \equiv 0 \text{ in (6)} \quad \text{for } i = k + 1, \dots, m. \quad (7)$$

With the aid of the maximum principle and Vyborny's theorem ⁽⁵⁾ one proves the uniqueness of the solution of the system (3)–(7).

Theorem 1. *Suppose that for problem (3)–(7) the following conditions are satisfied: 1) all coefficients and functions entering (3)–(6) are continuous in their domains of definition, and*

$$c_k^{(kl)}(x, t) \leq c < +\infty, \quad (x, t) \in \overline{D}_T^{(l)}, \quad k = 1, 2, \dots, m; \quad l = 1, 2;$$

2) equations (3) are uniformly parabolic in $\overline{D}_T^{(l)}$ with parabolicity constant $M_0 > 0$.

Then the solution of problem (3)–(7) is unique in the class of functions $u_{kl}(x, t)$, $k = 1, 2, \dots, m$; $l = 1, 2$, satisfying (3)–(7) and continuous on $\overline{D}_T^{(l)}$.

If one somewhat narrows the class of systems (3)–(7) by the conditions

$$b_{ij}^{(kl)}(x, t) \equiv 0 \text{ in (3)}; \quad a_i^{(k)}(x, t) \equiv 0 \text{ in (5)}; \quad d_{il}^{(k)}(x, t) \equiv 0 \text{ in (6)} \quad (8)$$

$$\text{for } i = 1, 2, \dots, k - 1,$$

then we obtain a class of systems possessing a number of interesting properties, following from the maximum principle, and allowing one, by means of the methods of the author and V. N. Maslennikova ⁽⁶⁾, to give, in particular, an a priori estimate for the maximum modulus of the solution of problem (3)–(8).

Theorem 2. Suppose that for the solution of problem (3)–(8) conditions 1), 2) of Theorem 1 are satisfied and, in addition: 3) the coefficients of equations (3) belong to the class $H^{0,\alpha,\alpha/2}(\overline{D}_T^{(l)})$ (see (3)), their maxima of moduli being bounded by a constant A , and, moreover,

$$c_k^{(kl)}(x, t) \leq c < 0, \quad (x, t) \in \overline{D}_T^{(l)};$$

4) the functions entering the right-hand sides of conditions (5), (6) are bounded in modulus by a constant B ; 5) the functions entering the left-hand sides of (5), (6) belong to the class $H_{1,\alpha,\alpha/2}^{0,1,(1+\alpha)/2}$; 6) the surfaces $\Gamma^{(l)}$ ($l = 1, 2$) are of class $\Pi_{1,1,(1+\alpha)/2}^{1,\alpha,\alpha/2}$, $0 < \alpha \leq 1$.

Then for the solution $u_{kl}(x, t)$ of problem (3)–(8), continuous on $\overline{D}_T^{(l)}$, the estimate

$$|u_{kl}|_0^{D_T^{(l)}} \leq C(d, d_0, \delta, a_0, A, B, c),$$

holds, where $C(\dots) > 0$ is a constant.

§ 2. Existence of a solution of (3)–(7).

Theorem 3. Suppose that for problem (3)–(7) conditions 1), 2) of Theorem 1 are satisfied. Suppose, in addition: 7) the surfaces $\Gamma^{(l)}$ ($l = 1, 2$) are of class $L_{1,1,(1+\beta)/2}^{1,\beta,\beta/2}$ ($0 < \alpha < \beta \leq 1$); 8) the coefficients in (3) have an (α) -norm (see (2⁴)) bounded by the constant M_1 ; 9) the initial functions from (4) have bounded $(2 + \alpha)$ -norms, and the coefficients entering into (5), (6) have $(1 + \alpha)$ -norms bounded by the constant M_2 ; 10) the initial conditions (4) and boundary conditions (5), (6) are compatible by virtue of (3) on the edges $\Gamma^{(s)} \cap \Omega^{(l)}$ ($l = 1, 2$ for $s = 1$ and $l = 2$ for $s = 2$).

Then there exists a solution $u_{kl}(x, t)$ ($k = 1, 2, \dots, m$; $l = 1, 2$) of problem (3)–(7) belonging to the class

$$H_{1,1,(1+\alpha)/2}^{1,\alpha,\alpha/2}(\overline{D}_T^{(l)}).$$

The proof of the existence of a solution of problem (3)–(7) is carried out by the classical method of continuation with respect to a parameter (see (7)), if one uses the $(2 + \alpha)$ -a priori estimate for the solution of problem (3)–(6), established by Theorem 4, and the existence of a solution from the class $H_{1,1,(1+\alpha)/2}^{1,\alpha,\alpha/2}(\overline{D}_T^{(l)})$ of the system (3⁰), (4)–(7), established by Theorem 5 (equation (3⁰) is obtained from (3) by putting in (3) $b_{ij}^{(kl)}(x, t) \equiv c_i^{(kl)}(x, t) \equiv 0$, $a_{ij}^{(kl)}(x, t) \equiv a_{ij}^{(kl)}$, where the matrix of constant coefficients $\|a_{ij}^{(kl)}\|$ is symmetric and positive definite).

Theorem 4. Suppose that all the conditions of Theorem 3 are satisfied (with 7) replaced by 6) from Theorem 2), and moreover

$$\det \|a_j^{(k)}(x, t)\| \neq 0, \quad (x, t) \in \Gamma^{(2)}, \quad \det \|d_{jl}^{(k)}(x, t)\| \neq 0, \quad (x, t) \in \Gamma^{(1)}, \quad l = 1, 2.$$

Suppose problem (3)–(6) has a solution $u_{kl}(x, t)$ for which

$$|u_{kl}|_{2+\alpha}^{D_T^{(l)}} < +\infty.$$

Then the estimate holds

$$\begin{aligned} |u_{kl}|_{2+\alpha}^{D_T^{(l)}} \leq & C(D_T^{(l)}, d, d_0, \delta, a_0, M_0, M_1, M_2) \max_{i,j} \left(|f_{ij}|_{\alpha}^{D_{T_j}^{(j)}} \right. \\ & \left. + |f_{ij}^{(1)}|_{2+\alpha}^{\Omega^{(j)}} + |f_j^{(2)}|_{1+\alpha}^{\Gamma^{(2)}} + |f_{ij}^{(3)}|_{1+\alpha}^{\Gamma^{(1)}} + |u_{ij}|_0^{D_T^{(j)}} \right). \end{aligned} \quad (9)$$

Remark. If uniqueness holds for problem (3)–(6) (see Theorems 1, 2), then, by virtue of the boundedness of D_T , in estimate (9) one may omit the term $|u_{ij}|_0^{D_T^{(j)}}$ appearing on the right.

Theorem 4 is proved by the same method by which, in the works of the author and V. N. Maslennikova⁽⁸⁾ (see also^(2,4)), a $(2 + \alpha)$ -a priori estimate was established for the solution of a problem with an oblique derivative for a parabolic equation of second order in a noncylindrical domain.

Theorem 5. Suppose that for problem (3⁰), (4)–(6), (8) all the conditions of Theorem 3 are satisfied. Then there exists a solution $u_{kl}(x, t)$ ($k = 1, 2, \dots, m$; $l = 1, 2$) of problem (3⁰), (4)–(7) belonging to the class

$$H_{1,1,(1+\alpha)/2}^{1,\alpha,\alpha/2}(\overline{D_T^{(l)}}).$$

The proof of Theorem 5 is carried out by the method of potentials with the aid of a number of results (see Lemmas 1–6) of the theory of special Pani heat potentials⁽⁹⁾, quite analogous to the theory of ordinary heat potentials from^(3,10,11). If on $\Gamma \equiv \Gamma^{(2)}$, the lateral boundary of the domain D_T , a field of directions $\nu(x, t)$ satisfying (7) is prescribed, and $P(\bar{x}, t; y, \tau)$ is the special fundamental solution of Pani^(9, §4), corresponding to $\nu(x, t)$ and to the parabolic operator with constant coefficients

$$\sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \frac{\partial}{\partial t},$$

then we consider the special heat surface potentials

potentials for $(\bar{x}, t) \in D_T$, $(x, t) \in \Gamma$

$$P[\varphi] \equiv P(\bar{x}, t) = \int_0^t d\tau \iint_{\Gamma_\tau} P(\bar{x}, t; y, \tau) \varphi(y, \tau) d\sigma_y(\tau), \quad (10)$$

$$Q[\varphi] \equiv Q(\bar{x}, t) = \frac{\partial P(\bar{x}, t)}{\partial \nu(x, t)}$$

and their direct values on Γ : $\bar{P}[\varphi] = \bar{P}(x, t)$, $\bar{Q}[\varphi] = \bar{Q}(x, t)$, obtained from (10) for $(\bar{x}, t) \equiv (x, t) \in \Gamma$.

Lemma 1. If Γ is of type ${}^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$ ($0 < \alpha \leq 1$), $[\varphi]_0^\Gamma < +\infty$, $[\nu]_0^\Gamma < +\infty$, then $P[\varphi] \in H^{0,\alpha_0,1/2}(\bar{D}_T)$ (α_0 is any number for which $0 < \alpha_0 < 1$), and the Hölder constants have the form $(C)[\varphi]_0^\Gamma$. Moreover, $P(\bar{x}, 0) \equiv 0$.

Lemma 2. If Γ is of type ${}^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$, $|\nu|_\beta^\Gamma < +\infty$ ($0 < \alpha \leq \beta \leq 1$) and $[\varphi]_0^\Gamma < +\infty$, then $\bar{Q}[\varphi] \in H^{0,\alpha^*,\alpha^*/2}(\Gamma)$ ($\alpha^* = \min(\alpha, \beta')$, where β' is any number for which $0 < \beta' < \beta$), and the Hölder constants have the form $(C)[\varphi]_0^\Gamma$, and $\bar{Q}(x, 0) \equiv 0$.

Lemma 3. Let Γ be of type ${}^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$, $|\nu|_\beta^\Gamma < +\infty$ ($0 < \alpha \leq \beta \leq 1$), and let $(\bar{x}, t) \in \Omega_t$ tend to x , $(x, t) \in \Gamma$, along the inner normal to Γ_t . Then for continuous φ

$$\lim_{(\bar{x}, t) \rightarrow (x, t) \in \Gamma_t} Q(\bar{x}, t) = \bar{Q}(x, t) - 2^{n-1} \pi^{n/2} (\det \|a^{ij}\|)^{-1/2} \varphi(x, t),$$

where $\|a^{ij}\|$ is the matrix inverse to $\|a_{ij}\|$. If $|\varphi|_\alpha^\Gamma < +\infty$ and $\varphi(x, 0) \equiv 0$,

$$|Q(\bar{x}, t) - \bar{Q}(x, t) + 2^{n-1} \pi^{n/2} (\det \|a^{ij}\|)^{-1/2} \varphi(x, t)| \leq (C) |\varphi|_\alpha^\Gamma |\bar{x} - x|^{\alpha^0}$$

$$(\alpha^0 = \alpha \text{ for } 0 < \alpha < 1; \alpha^0 = \alpha_0 \text{ for } \alpha = 1).$$

Lemma 4. Let Γ be of type ${}^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}$, $|\nu|_{1+\beta}^\Gamma < +\infty$, $|\varphi|_\alpha^\Gamma < +\infty$ ($0 < \alpha \leq \beta \leq 1$), and $\varphi(x, 0) \equiv 0$. Then $Q[\varphi] \in H^{0,1,(1+\alpha^*)/2}_{1,\alpha',\alpha'/2}(\Gamma)$, and the Hölder constants have the form $(C)|\varphi|_\alpha^\Gamma$, and $\bar{Q}(x, 0) \equiv 0$.

Lemma 5. Let Γ be of type ${}^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}$, $|\nu|_\beta^\Gamma < +\infty$, $|\varphi|_\alpha^\Gamma < +\infty$ ($0 < \alpha \leq \beta \leq 1$), and $\varphi(x, 0) \equiv 0$. Then $P[\varphi] \in H^{0,1,(1+\alpha')/2}_{1,\alpha',\alpha'/2}(\bar{D}_T)$, and the Hölder constants have the form $(C)|\varphi|_\alpha^\Gamma$, and $P(x, 0) \equiv 0$.

Lemma 6. Let Γ be of type ${}^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}$, $|\nu|_{1+\beta}^\Gamma < +\infty$, $|\varphi|_{1+\alpha}^\Gamma < +\infty$, $\varphi(x, 0) \equiv 0$ ($0 < \alpha \leq \beta \leq 1$). Then $P[\varphi] \in H^{1,\alpha',\alpha'/2}_{1,1,(1+\alpha')/2}(\bar{D}_T)$, and the Hölder constants have the form $(C)|\varphi|_{1+\alpha}^\Gamma$, and $P(x, 0) \equiv 0$.

Remark. If Γ is of type ${}^{1,\beta,\beta/2}_{1,1,(1+\beta)/2}$, where $0 < \alpha < \beta \leq 1$, then, under the remaining assumptions of Lemma 6, $P[\varphi] \in H^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}(\bar{D}_T)$.

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1. V. I. Skobelkin, A. A. Boldin, DAN, 145, No. 6, 1396 (1962).
2. L. I. Kamynin, V. N. Maslennikova, DAN, 153, No. 3, 526 (1963).
3. L. I. Kamynin, DAN, 160, No. 2, 271 (1965).
4. L. I. Kamynin, V. N. Maslennikova, DAN, 160, No. 3, 527 (1965).
5. R. Vyborny, DAN, 117, No. 4, 563 (1957).
6. L. I. Kamynin, V. N. Maslennikova, Siberian Math. Journal, 2, No. 3, 384 (1961).
7. S. N. Bernstein, Math. Ann., 69, 82 (1910).
8. L. I. Kamynin, V. N. Maslennikova, Siberian Math. Journal, 7, No. 1 (1966).
9. M. Ragni, Ann. Scuola norm. sup. Pisa, ser. III, II, fasc. I–II, 73 (1957).
10. L. I. Kamynin, Differential Equations, 1, No. 6, 799 (1965).
11. L. I. Kamynin, Differential Equations, 2, No. 5 (1966).

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