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Abstract

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MATHEMATICS

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ON THE SOLVABILITY OF THE BASIC BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF PARABOLIC TYPE WITH DEGENERATING PRINCIPAL PART

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Let $\Omega = G \times [0, t_0]$ be a cylinder in the Euclidean space of variables x, y, t . Suppose that the boundary Γ of the plane domain G consists of the segment $\delta : y = 0, a \leq x \leq b$, and a Jordan curve σ with endpoints at the points $(a, 0)$ and $(b, 0)$, lying in the half-plane $y > 0$. We shall say that the domain G belongs to the class A_1 if its boundary Γ has everywhere nonnegative piecewise-continuous curvature. Denote by $W_{2(y)}^{2,1}(\Omega)$ the space of functions $u(x, y, t)$ summable in the domain Ω , having generalized derivatives of first order with respect to t and of first and second orders with respect to x and y , with finite norm

$$\|u\|_{W_{2(y)}^{2,1}}^2 = \int_{\Omega} (y^2 u_{xx}^2 + y u_{xy}^2 + u_{yy}^2 + y u_x^2 + u_y^2 + u_t^2 + u^2) d\Omega. \quad (1)$$

Similarly, we introduce the space $W_{2(y)}^2(\Omega)$ of functions $u(x, y)$, $(x, y) \in G$, with finite norm

$$\|u\|_{W_{2(y)}^2}^2 = \int_G (y^2 u_{xx}^2 + y u_{xy}^2 + u_{yy}^2 + y u_x^2 + u_y^2 + u^2) dG. \quad (2)$$

The spaces $W_{2(y)}^{2,1}$ and $W_{2(y)}^2$ are complete. Indeed, owing to the boundedness of the domain G , the inequality

$$\|u\|_{W_{2(y,2,2,2)}^2}^2 = \int_G y^2 (u_{xx}^2 + u_{xy}^2 + u_{yy}^2 + u_x^2 + u_y^2 + u^2) dG \leq c \|u\|_{W_{2(y)}^2}^2, \quad (3)$$

holds, where $W_{2(y,2,2,2)}^2$ is the complete space ⁽¹⁾.

By virtue of (3), every fundamental sequence $\{u_n\}$ in $W_{2(y)}^2$ will converge to a function $v(x, y)$ having generalized derivatives up to second order, and in order to prove completeness of $W_{2(y)}^2(G)$ it is enough to show that $v(x, y) \in W_{2(y)}^2(G)$.

By the fundamentalness of the sequence $\{u_n\}$ in $W_{2(y)}^2$, the inequality

$$\int_G y [(u_n - u_k)_{xy}]^2 dG < \varepsilon,$$

holds, i.e.

$$\int_G y (u_{nxy} - u_{kxy})^2 dG < \varepsilon.$$

Hence it follows that $\{u_{nxy}\}$ is fundamental in $W_{2(y,1)}^0$. Owing to the completeness of $W_{2(y,1)}^0$, there exists a function $\omega \in W_{2(y,1)}^0$ such that

$$\int_G y (u_{nxy} - \omega)^2 dG < \varepsilon,$$

but then also

$$\int_G y^2 (u_{nxy} - \omega)^2 dG < \varepsilon,$$

i.e. $\{u_{nxy}\}$ converges to ω in $W_{2(y,2)}^0$. Taking into account the convergence of $\{u_{nxy}\}$ to v_{xy} in the space $W_{2(y,2)}^0$,

we conclude that v_{xy} coincides with ω almost everywhere. Thus,

$$\int_G y v_{xy}^2 dG$$

exists.

The existence of the other integrals on the right-hand side of (2) in the expression $\|v\|_{W_{2(y)}^2}$ is proved analogously.

Now note that functions from $W_{2(y)}^2$ belong to the space $W_{2(y,1,0)}^1 \Omega$, and, moreover, in each section $t = \text{const}$ of the domain Ω they may be regarded as functions of the space $W_{2(y)}^2$. Repeating the reasoning given above, one easily verifies the validity of the second part of our assertion.

The main boundary-value problem. Determine in Ω a solution of the equation

$$Lu \equiv Tu - u_t \equiv y u_{xx} + u_{yy} - u_t = f(x, y, t) \quad (4)$$

under the condition

$$u|_{G \cup S} = 0, \quad (5)$$

where S is the lateral surface of the cylinder Ω .

For an equation of parabolic type in the absence of degeneration of the principal part, the problem with boundary condition (5) in the class of functions $u \in W_2^2(\Omega)$ with respect to x, y and $W_2^1(\Omega)$ with respect to t was solved in ⁽²⁾. For degenerating parabolic equations with a sufficiently smooth right-hand side in domains with smooth boundary, an analogous problem was considered in ⁽³⁾.

Denote by $W_{2,0(y)}^2$ the space of functions $u(x, y)$ obtained by completing, in the norm $W_{2(y)}^2$, the functions three times continuously differentiable in $\overline{G}$ and vanishing on Γ . On the basis of the obvious inequality

$$\|u\|_{W_2^2(G_\varepsilon)} \leq c\|u\|_{W_{2(y)}^2(G)},$$

where $G_\varepsilon \subset G$ is the domain obtained by moving a distance $\varepsilon > 0$ away from the x -axis, by the Sobolev embedding theorem ⁽⁴⁾, the functions $u(x, y) \in W_{2,0(y)}^2(G)$ are continuous in $\overline{G_\varepsilon}$ and vanish on $\sigma \cap \overline{G_\varepsilon}$.

The inequality ⁽⁵⁾ holds:

$$\int_\delta u^2 dx \leq c \int_G (yu_x^2 + u_y^2 + u^2) dG$$

for functions vanishing on Γ , from which it follows that functions from $W_{2,0(y)}^2(G)$ vanish on the segment δ in the mean.

The space $W_{2,0(y)}^{2,1}$ of functions $u(x, y, t)$ is introduced analogously. Functions $u \in W_{2,0(y)}^{2,1}$ also vanish on $(G \cup S)$ in the mean. This follows from the fact that, for fixed t , they may be regarded as functions from $W_{2,0(y)}^2$, and, moreover,

$$\int_G t_0 u^2(x, y, t) dG = \int_\Omega [u^2 - 2(t_0 - t)uu_t] d\Omega \leq c\|u\|_{W_{2(y)}^{2,1}\Omega}.$$

Let us first consider the homogeneous Dirichlet problem for the equation $Tu = h(x, y)$ in the domain G .

Lemma 1. If $u \in W_{2,0(y)}^2(G)$, then

$$\|Tu\|_{L_2(G)} \geq c\|u\|_{W_{2(y)}^2(G)}. \quad (6)$$

It is sufficient to carry out the proof for functions having continuous derivatives up to the third order in $\overline{G}$ and vanishing on Γ , and then, by means of a limiting passage, to extend it to all functions from $W_{2,0(y)}^2(G)$.

Using Hölder's inequality, from Green's formula, by virtue of the condition

$$u|_{\Gamma} = 0, \quad (7)$$

we obtain

$$c_1 \int_G u^2 dG \leq c \int_G (yu_x^2 + u_y^2) dG \leq \left[\int_G u^2 dG \right]^{1/2} \left[\int_G (yu_{xx} + u_{yy})^2 dG \right]^{1/2}, \quad (8)$$

whence

$$\int_G (yu_x^2 + u_y^2 + u^2) dG \leq c \int_G (Tu)^2 dG.$$

Let $x = x(s)$, $y = y(s)$ ($0 \leq s \leq s_0$, $y(0) = 0$, $x(0) = b$, $y(s_0) = 0$, $x(s_0) = a$) be the parametric equation of σ . We divide the curve σ into a finite number of arcs σ_i in such a way that on $\sigma_1, \sigma_2, \dots, \sigma_k$ the inequality $x'_s \neq 0$ holds, and on $\sigma_{k+1}, \sigma_{k+2}, \dots, \sigma_{k+r}$ the inequality $y'_s \neq 0$ holds. In view of the fact that, by (7), $du/ds|_{\sigma} = d^2u/ds^2|_{\sigma} = 0$, on the basis of the obvious identity

$$\begin{aligned} 2 \int_G yu_{xx}u_{yy} dG &= \int_{\sigma} (yu_xu_{yy} - yu_yu_{xy} - u_xu_y) dy + \\ &+ (yu_xu_{xy} - yu_yu_{xx} - u_x^2) dx + 2 \int_G yu_{xy}^2 dG, \end{aligned}$$

we shall have

$$\begin{aligned} 2 \int_G yu_{xx}u_{yy} dG &= \int_{\sigma_i} \sum_{i=1}^k \frac{y(x'y'' - x''y')}{(x')^2} u_y^2 ds + \\ &+ \int_{\sigma_i} \sum_{i=k+1}^{k+r} \frac{y(x'y'' - x''y')}{(y')^2} u_x^2 ds + 2 \int_G yu_{xy}^2 dG. \end{aligned} \quad (9)$$

As a consequence of the nonnegativity of the curvature of σ_i , from (9) we conclude

$$\int_G yu_{xy}^2 dG \leq \int_G yu_{xx}u_{yy} dG, \quad (10)$$

whence the validity of Lemma 1 follows.

We shall say that the domain G belongs to the class A_2 if the homogeneous Dirichlet problem for the Poisson equation $\Delta u = h(x, y)$ is solvable in $W_{2,0}^2(G)$ (5) for every right-hand side h from $L_2(G)$.

Theorem 1. If the domain G belongs to $A_1 \cap A_2$, then the homogeneous Dirichlet problem for the equation

$$Tu = f(x, y) \quad (11)$$

is uniquely solvable in $W_{2,0(y)}^2$ for any $f \in L_2(G)$.

The range of the operator T , defined on functions from $W_{2,0(y)}^2$, by Lemma 1, is closed in L_2 . Consequently, to prove Theorem 1 it remains to show that the following is true.

Lemma 2. If $v \in L_2(G)$ and

$$\int_G vTu \, dG = 0 \quad (12)$$

for all $u \in W_{2,0(y)}^2$, then $v \equiv 0$.

Since the domain G belongs to the class A_2 , there exists a function $\bar{u} \in W_{2,0}^2(G) \subset W_{2,0(y)}^2(G)$ such that $\Delta \bar{u} = v$. Substituting in (12), instead of u and v , respectively $u = \bar{u}$, $v = \Delta \bar{u}$, we obtain

$$\int_G \Delta \bar{u} T\bar{u} \, dG = \int_G [y\bar{u}_{xx}^2 + \bar{u}_{yy}^2 + (1+y)\bar{u}_{xx}\bar{u}_{yy}] \, dG.$$

For the function $\bar{u}$ there is an inequality analogous to (10):

$$\int_G (1+y)\bar{u}_{xx}\bar{u}_{yy} \, dG \geq \int_G (1+y)\bar{u}_{xy}^2 \, dG,$$

therefore,

$$c\|\bar{u}\|_{W_{2(y)}^2} \leq c \int_G (T\bar{u})^2 \, dG \leq \int_G (y\bar{u}_{xx}^2 + \bar{u}_{yy}^2 + \bar{u}_{xy}^2) \, dG \leq \int_G \Delta \bar{u} T\bar{u} \, dG = 0,$$

i.e. $\bar{u} \equiv v \equiv 0$.

The uniqueness of the solution of problem (11)–(7) follows from Lemma 1.

We now return to the main problem for equation (4).

Lemma 3. If $u \in W_{2,0(y)}^{2,1}(\Omega)$, then

$$\|u\|_{W_{2(y)}^{2,1}(\Omega)} \leq c \|Lu\|_{L_2(\Omega)}.$$

The proof is carried out in the same way as in work ² in establishing an analogous inequality in the case of the heat equation.

Theorem 2. If the domain G belongs to $A_1 \cap A_2$, then the main boundary-value problem (5) for equation (4) is uniquely solvable in $W_{2,0(y)}^{2,1}$ for any right-hand side from $L_2(\Omega)$.

Uniqueness follows from Lemma 3.

Solvability of problem (4)–(5) is proved by means of a density lemma analogous to Lemma 2. In proving this lemma, as $u(x, y, t) \in W_{2,0(y)}^{2,1}(\Omega)$ one takes the solution of the equation $c_1 u_{xx} + u_{yy} - u_t = v$, $c_1 = \max y$, $(x, y) \in G$, satisfying the boundary condition (5).

We now consider the equation

$$L_1 u \equiv Lu + au_x + bu_y + cu = f(x, y, t), \quad (13)$$

where $a_x, b_y, c \in C^{(0,0)}(\Omega)$, $f \in L_2(\Omega)$.

Theorem 3. If the domain G belongs to $A_1 \cap A_2$ and $a = o(1/2)$, $-\frac{1}{2}a_x + \frac{1}{2}b_y - c \geq \nu > -\frac{1}{4}y_{\max}$ in Ω , then the main boundary-value problem (6) for equation (13) is uniquely solvable in $W_{2,0(y)}^{2,1}(\Omega)$ for any right-hand side f from $L_2(\Omega)$.

The proof of Theorem 3 is built on the basis of Theorem 2 by applying the known method of continuation with respect to a parameter.

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CITED LITERATURE

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