

ON THE APPROXIMATION AND CONVERGENCE OF THE GRID METHOD IN ELLIPTIC PROBLEMS

Yu. K. Dem'yanovich

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.26925>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.949.8

MATHEMATICS

Yu. K. DEM' YANOVICH

ON THE APPROXIMATION AND CONVERGENCE OF THE GRID METHOD IN ELLIPTIC PROBLEMS

(Presented by Academician V. I. Smirnov on 6 XII 1965)

In papers ^(1-5, 20) projection methods for constructing difference equations were considered. In particular, in the case of the Dirichlet problem in a domain Ω with twice continuously differentiable boundary, convergence ^(3, 5) of the grid method in $\dot{W}_2^1(\Omega)$ with rate $h^{1/2}$ was established. Using R. Courant's proposed ⁽¹⁾ linear extension of a grid function over triangles, in the two-dimensional case for the third boundary-value problem L. A. Oganessian constructed ⁽²⁰⁾ equations of the grid method with convergence rate h in $W_2^1(\Omega)$ and condition number of order h^{-3} .

The purpose of the present paper is to obtain, in the n -dimensional case, a theorem on the approximation of functions of the class $W_2^1(\Omega)$ by functions obtained by linear extension over simplices of grid functions, and, in the case of the Dirichlet and Neumann problems, to construct stable ⁽⁵⁾ schemes of the grid method converging with rate h in $W_2^1(\Omega)$, and also to consider some properties of the indicated schemes. Below we shall use the notation of note ⁽³⁾.

1°. Theorem 1. *For any finite n -dimensional domain Ω with twice continuously differentiable boundary S , there exists a family $\{R_h\}$ of simplicial complexes of the form $R_h = P_h \cup Q_h$, $\text{mes } |P_h \cap Q_h| = 0$, where Q_h is an arbitrary simplicial subdivision of the cellular complex whose cells are cubes of a grid with step h , and P_h is a simplicial complex lying in a boundary strip of width $c_0 h$, such that: 1) the body $|R_h|$ of the complex R_h lies in Ω ; 2) the set $\Omega \setminus |R_h|$ lies inside a strip of width $c_1 h^2$; 3) the length l of any edge and the value α of any angle between edges with a common endpoint of the complex R_h satisfy the inequalities $c_2 h \leq l \leq c_3 h$, $\alpha_0 \leq \alpha \leq \alpha_1$, where $c_0, c_1, c_2, c_3, \alpha_0$, and α_1 are positive numbers independent of the choice of edges and angles in R_h , and also of the step h , with $0 < \alpha_0 \leq \alpha_1 < \pi$.*

We shall say that a function ψv is obtained by linear extension over simplices of a grid function v , given on the zero-dimensional skeleton of the complex R_h , if for each of its simplices T

$$(\psi v)(x) = \sum_{i=0}^n \lambda^i v(x_i), \quad x \in T,$$

where $x_0, \dots, x_n$ are the vertices of the simplex T , and $\lambda^0, \dots, \lambda^n$ are the barycentric coordinates of the point x in this simplex. In what follows $\Omega_{(\varepsilon)}$ denotes the enlargement of the domain Ω to its ε -neighborhood, and $\omega_{W_2^1(\Omega_{(\varepsilon)})}(g, h)$ the modulus of continuity of the function g in the class $W_2^1(\Omega_{(\varepsilon)})$.

Theorem 2. *Let in the domain $\Omega_{(\varepsilon)}$ there be given a family $\{R_h\}$ of simplicial complexes containing the domain Ω , which satisfy condition 3) of the preceding theorem. Let the function g belong to $W_2^1(\Omega_{(\varepsilon)})$. Then for each h ($h < \varepsilon/2c$) there exists a function defined on the zero-dimensional skeleton of the com-*

plex R_h , a mesh function v , whose linear extension ψv over the simplices of the complex R_h approximates the function g in such a way that

$$\|g - \psi v\|_{W_2^1(|R_h|)} \leq K \omega_{W_2^1(\Omega_{\varepsilon/2})}(g, ah),$$

where a and K are determined by the numbers c, a_0 and a_1 and do not depend on g and h .

Let now $g \in \overset{\circ}{W}_2^1(\Omega)$, and suppose that the boundary S of the domain Ω is three times continuously differentiable. Choose ε so small that the points $x \in \Omega_{(\varepsilon)} \setminus \Omega$ admit a locally unique representation $x = y + \tau m(y)$, $\tau \geq 0$, where $m(y)$ is the exterior normal to the surface S at the point y , while the point $x' = y - \tau m(y)$ locally uniquely determines y and τ . In accordance with the results of V. M. Babich ⁽⁶⁾ and S. M. Nikol'skii ⁽⁷⁾, the function g can be extended to $\Omega_{(\varepsilon)}$ with preservation of the class and of the Hölder exponent by the formula

$$g(y + \tau m(y)) = -g(y - \tau m(y)), \quad y \in S. \quad (1)$$

Theorem 3. Let the domain Ω have a three-times continuously differentiable boundary. Let the function $g \in \overset{\circ}{W}_2^1(\Omega)$ be extended to the domain $\Omega_{(\varepsilon)}$ by formula (1). Suppose, moreover, that a family $\{R_h\}$ of simplicial complexes satisfying the conditions of Theorem 1 is given. Then for each h there exists a mesh function v , defined on the zero-dimensional skeleton of the complex R_h and equal to zero on the boundary of the set $|R_h|$, whose linear extension $\psi_0 v$ over the simplices of R_h , continued by zero to the whole set Ω , approximates the function g in such a way that, for $h < \varepsilon/2c$,

$$\begin{aligned} \|g - \psi_0 v\|_{W_2^1(\Omega)} &\leq K \omega_{W_2^1(\Omega_{\varepsilon/2})}(g, ah) + C_1 h^{1/2} \omega_{W_2^1(\Omega)}(g', a_1 h) + \\ &+ C_2 \omega_{W_2^1(\Omega)}(g', a_2 h^2), \end{aligned}$$

where the constants K, C_1, C_2, a, a_1 , and a_2 do not depend on g and h , and by g' is denoted the extension of the function g by zero to the exterior of the domain Ω .

2°. Let us apply Theorems 1-3 to the grid method for the Dirichlet problem

$$A_0 u \equiv - \sum_{i,k=1}^n \frac{\partial}{\partial x_i} \left(a_{ik} \frac{\partial u}{\partial x_k} \right) + au = f, \quad u|_S = 0, \quad (2)$$

understood in the same way as in (8). With respect to the functions a_{ik}, a , and f , the assumptions of the note (3) are retained.

Theorem 4. Let the boundary S of the domain Ω be three times continuously differentiable, and let the assumptions of the preceding theorem and of Theorem 3 of the note (3) be fulfilled. For the function v_* , solving the problem of minimizing the functional

$$F(\psi_0 v) = (A_0^{1/2} \psi_0 v, A_0^{1/2} \psi_0 v) - 2(f, \psi_0 v)$$

in the space $\bar{X}$ of mesh functions v equal to zero on the boundary of the set $|R_h|$, the estimate

$$\|u^* - \psi_0 v_*\|_{W_2^1(\Omega)} \leq Ch \|f\|_{L_2(\Omega)}$$

is valid, where the positive constant C depends only on the data of the exact problem (2), and u^* denotes the solution of problem (2).

An analogous assertion is valid for the Neumann problem, and also in the case of the Dirichlet problem for a strongly elliptic system of second-order differential equations.

Remark. Under the conditions of Theorem 4 and under the condition that the family $\{R_h\}$ satisfies the assumptions of Theorem 1, the grid method is stable in the sense of the definition formulated in (5). In the one-dimensional case, the subspaces given in (5) coincide with the subspaces constructed earlier by A. A. Samarskii (9).

3°. Let us note that the difference schemes obtained here, as well as in the works (3, 5), fit into the general theory of difference schemes developed by A. N. Tikhonov and A. A. Samarskii (9-16). In what follows we shall use the concepts of this theory (16).

Define the operator $\bar{A}_0$ by the equality

$$(\bar{A}_0 v, v_1)_0 = (A_0^{1/2} \psi_0 v, A_0^{1/2} \psi_0 v_1), \quad v, v_1 \in \bar{X}, \quad (3)$$

where the brackets $(\dots)_0$ denote the scalar product in the space $\overline{X}$, introduced in (5). The operator $\overline{A}_0$ has the form

$$\overline{A}_0 v(x_0) = \sum_{y_0 \in \mathfrak{M}(x_0)} A_{x_0 y_0} v(y_0), \quad (4)$$

where $A_{x_0 y_0}$ are functionals on the coefficients of the operator A_0 , and $\mathfrak{M}(x_0)$ is a certain set of nodes, called the mesh stencil.

Scheme (4) is called ⁽¹⁶⁾ homogeneous if $A_{x_0 y_0} = A_{y_0}(\mathfrak{A}_0)$, where $\mathfrak{A}_0$ is the vector of coefficients of the operator A_0 , transformed by means of the mapping χ_{x_0}

$$(\chi_{x_0} \varphi)(s) = \varphi(x_0 + sh), \quad s \in \mathfrak{Z}_0;$$

$\mathfrak{Z}_0$ is a certain set of n -dimensional vectors independent of h , called the reduced coefficient stencil; A_{y_0} is a functional on the space K_0 of vector-functions defined on $\mathfrak{Z}_0$; $\mathfrak{M}(x_0)$ is the set of vectors of the form $y = x_0 + sh$, $s \in \mathfrak{M}_0$; here $\mathfrak{M}_0$ is the reduced mesh stencil.

Theorem 5. *Let the mesh be regular, i.e. let the stars of the complex with vertices inside the polyhedron R_h be congruent to one another. Then the mesh scheme (3) is homogeneous, conservative, and linear, with stencil functionals of rank ∞ .*

Let the mapping χ be given on the set of functions continuous in Ω by the formula $(\chi u)(x_0) = u(x_0)$, $u \in C(\Omega)$.

There are known ^(9–18) schemes with a positive definite operator which converge in the class of discontinuous coefficients and have, on smooth functions, first-order approximation. There is an example ^(16, 17) of schemes which do not converge in the class of discontinuous coefficients but have second-order approximation. For the schemes considered here the following is established.

Theorem 6. *Let, in some subdomain Ω' of the domain Ω , the mesh be regular. Then for the mesh scheme (3), for $u \in C_4(\Omega')$, the representation holds*

$$(\overline{A}_0 \chi u)(x_0) = (\chi A_0 u)(x_0) + \alpha(h), \quad (5)$$

where

$$\alpha(h) \rightarrow 0, \quad \text{as } h \rightarrow 0, \quad \text{if } a_{ik} \in C_1(\Omega'), \quad a \in C(\Omega'), \quad (6)$$

$$\alpha(h) = O(h), \quad \text{if } a_{ik} \in C_2(\Omega'), \quad a \in C_1(\Omega'), \quad (7)$$

$$\alpha(h) = O(h^2), \quad \text{if } a_{ik} \in C_3(\Omega'), \quad a \in C_2(\Omega'), \quad (8)$$

and the asymptotics indicated here for $\alpha(h)$ are uniform with respect to $x_0 \in \Omega'$.

Remark 1. A statement analogous to Theorem 6 is valid for the schemes given in ⁽³⁾.

Remark 2. In the case of families of meshes that are not regular, only the estimates (6), (7) hold.

For a smooth solution u^* of problem (2), the fact of approximation for $n = 1$ implies convergence of the mesh method in the metric $C(\Omega)$, with the order of approximation. For $n \geq 2$ the same assertion is valid in the case where the maximum principle holds.

Theorem 7. Let the internal normals to the $(n-1)$ -dimensional faces of each simplex T of the complex R_h form nonobtuse angles in the sense of the scalar product

$$(\xi, \eta)_T = \sum_{i,j=1}^n \left(\int_T a_{ij} dx \right) \xi_i \eta_j;$$

where $\xi = (\xi_1, \dots, \xi_n)$, $\eta = (\eta_1, \dots, \eta_n)$ are n -dimensional vectors. Then, for the operator A_0 constructed here, the maximum principle is satisfied.

Remark. For $n = 2$, the fulfillment of the conditions of Kh. L. Smolitskii ((19), p. 98) implies the validity of the conditions of Theorem 7. In the case of a mesh consisting of right triangles with one of the sides parallel to the x_1 -axis, from Theorem 7 one easily obtains the following simple sufficient conditions for the maximum principle to hold:

$$3a_{11} \geq a_{22}, \quad |a_{12}| \leq a_{22}/\sqrt{3}.$$

In conclusion I express my deep gratitude to Prof. S. G. Mikhlin for his attention to this work.

Leningrad State University
named after A. A. Zhdanov

Received
2 XII 1965

CITED LITERATURE

¹ R. Courant, Bull. Am. Math. Soc., **49**, No. 1, 1 (1943). ² L. A. Oganessian, *Solution of Engineering Problems on a Computer*, L., 1963. ³ Yu. K. Demyanovich, DAN, **159**, No. 2 (1964). ⁴ Yu. A. Gusman, L. A. Oganessian, *Zhurn. vychislit.*

matem. i matem. fiz., **5**, No. 2 (1965). ⁵ Yu. K. Demyanovich, DAN, **164**, No. 1 (1965). ⁶ V. M. Babich, UMN, **8**, issue 2 (64) (1953). ⁷ S. M. Nikol'skii, *Mat. sbornik*, **40** (82), No. 2 (1956). ⁸ S. G. Mikhlin, *The Problem of the Minimum of a Quadratic Functional*, 1952. ⁹ A. N. Tikhonov, A. A. Samarskii, DAN, **131**, No. 6 (1960). ¹⁰ A. N. Tikhonov, A. A. Samarskii, DAN, **108**, No. 3 (1956). ¹¹ A. N. Tikhonov, A. A. Samarskii, DAN, **122**, No. 4 (1958). ¹² A. N. Tikhonov, A. A. Samarskii, DAN, **124**, No. 3 (1959). ¹³ A. N. Tikhonov, A. A. Samarskii, DAN, **124**, No. 4 (1959). ¹⁴ A. N. Tikhonov, A. A. Samarskii, DAN, **131**, No. 3 (1960). ¹⁵ A. N. Tikhonov, A. A. Samarskii, DAN, **131**, No. 3 (1960). ¹⁶ A. N. Tikhonov, A. A. Samarskii, *Zhurn. vychislit. matem. i matem. fiz.*, **1**, No. 1 (1961). ¹⁷ A. N. Tikhonov, A. A. Samarskii, DAN, **149**, No. 3 (1963). ¹⁸ O. A. Ladyzhenskaya, *Mixed Problem for a Hyperbolic Equation*, 1953. ¹⁹ S. G. Mikhlin, Kh. L. Smolitskii, *Approximate Methods for Solving Differential and Integral Equations*, 1965. ²⁰ L. A. Oganessian, DAN, **170**, No. 1 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.