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Abstract

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MATHEMATICS

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ON PSEUDODIFFERENTIAL OPERATORS OF PRINCIPAL TYPE

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Consider a smooth manifold Ω , on which two complex vector bundles E and F with infinitely differentiable structure are defined. Following L. Hörmander ⁽¹⁾, we shall call a continuous mapping P from E to F a **pseudodifferential operator** if, from the space of finite sections $C_0^\infty(\Omega, E)$ into $C^\infty(\Omega, F)$, the function $e^{-i\tau\varphi}P(ue^{i\tau\varphi})$ admits an asymptotic expansion in powers of τ as $\tau \rightarrow \infty$, provided $u \in C_0^\infty(\Omega, E)$, $\varphi \in C^\infty(\Omega)$, and $\text{grad } \varphi \neq 0$ on $\text{supp } u$. The **symbol** of the operator P is the formal sum $\sum_0^\infty p^j(x, \xi)$, coinciding with the asymptotic series for $e^{-i(x, \xi)}P(ue^{i(x, \xi)})$ as $\xi \rightarrow \infty$. In ⁽¹⁾ it is shown that the principal part of the symbol $p^0(x, \xi)$ is invariantly defined on the cotangent space to Ω .

As is known, the operator P is called **elliptic** if $p^0(x, \xi)$ effects an invertible mapping: $E_x \rightarrow F_x$, when $x \in \Omega$, and $\xi \neq 0$ is a vector cotangent to Ω at the point x . It is known that an operator P of order m is elliptic if and only if

$$\|u\|_{s+m} \leq C(\|Pu\|_s + \|u\|_t), \quad u \in C_0^\infty(K, E),$$

where K is an arbitrary compact set in Ω ; s and t are arbitrary real numbers:

$$\|u\|_s = \left(\int |\tilde{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi \right)^{1/2}.$$

Here

$$\tilde{u}(\xi) = \int u(x) e^{-i(x, \xi)} dx$$

is the Fourier transform of the function $u(x)$. In ⁽²⁾ L. Hörmander showed that the existence of the estimate

$$\|u\|_s \leq C(\|Pu\|_{s-m+\delta} + \|u\|_t), \quad u \in C_0^\infty(K, E) \quad (1)$$

for $0 < \delta < 1/2$ is also equivalent to ellipticity of the operator P , but for $\delta = 1/2$ a new class of operators arises, called **subelliptic** by Hörmander. In the present work we describe pseudodifferential operators for which estimate (1) holds with $0 \leq \delta < 1$. These operators should naturally be called **operators of principal type**, since for them (and only for them) the existence of an estimate of this form is determined solely by the principal part $p^0(x, \xi)$ of the symbol of the operator P .

Theorem 1. *The estimate (1) with $\delta = k/(k+1)$ holds if and only if, for every compact subset $K \subset \Omega$, there exists a constant C such that, for $x \in K$ and $0 \neq \xi \in R^n$,*

$$\int |\psi(y)|^2 dy \leq C \int \left| \sum_{|\alpha|+|\beta| \leq k} \frac{1}{\alpha! \beta!} \frac{\partial^{\alpha+\beta} q(x, \xi)}{\partial \xi^\alpha \partial x^\beta} |\xi|^{k(|\alpha|-|\beta|)/(k+1)} y^\beta D^\alpha \psi(y) \right|^2 dy +$$

$$+ \varepsilon(\xi) \sum_{|\alpha|+|\beta| \leq k+2} |\xi|^{-2|\alpha|(k-1)/(k+1)} \int |y^\beta D^\alpha \psi(y)|^2 dy, \quad \psi \in C_0^\infty(R^n)^e. \quad (2)$$

Here

$$D = \left(\frac{1}{i} \frac{\partial}{\partial x_1}, \dots, \frac{1}{i} \frac{\partial}{\partial x_n} \right);$$

e is the dimension of the bundle E ; $q(x, \xi) = p^0(x, \xi) |\xi|^{-m+k/(k+1)}$; m is the order of the operator P ; $\varepsilon(\xi) \rightarrow 0$ as $\xi \rightarrow \infty$.

Apparently, the operators described by Theorem 1 exhaust all operators of principal type.

Theorem 2. If, for the operator P , estimate (1) holds with $0 < \delta < 2/3$, and $e = f = 1$, then this estimate also holds for $\delta = 1/2$.

In applications the following variant of Theorem 1 is convenient, in which it is not required that the operator have order $k/(k+1)$.

Theorem 1'. For an operator P of order m , estimate (1) holds if and only if, for every compact subset $K \subset \Omega$, there exist a constant C and a function $\varepsilon(\xi)$, tending to zero as $\xi \rightarrow \infty$, such that

$$\int |\psi(y)|^2 dy \leq C |\xi|^{-2m+2k/(k+1)} \int \left| \sum_{|\alpha|+|\beta| \leq k} \frac{\partial^{\alpha+\beta} p^0(x, \xi)}{\partial \xi^\alpha \partial x^\beta} \frac{1}{\alpha! \beta!} \times \right.$$

$$\left. \times |\xi|^{k(|\alpha|-|\beta|)/(k+1)} y^\beta D^\alpha \psi(y) \right|^2 dy +$$

$$+\varepsilon(\xi) \int \sum_{|\alpha|+|\beta|\leq k+2} |\xi|^{-2|\alpha|(k-1)/(k+1)} |y^\beta D^\alpha \psi(y)|^2 dy, \quad \psi \in C_0^\infty(R^n). \quad (2')$$

Of course, conditions (2) and (2') are not always easy to verify. However, they make it possible to reduce the difficult question of estimates of type (1) for a pseudodifferential operator (including for a differential operator of arbitrary order) to the question of the existence of the simpler estimate (2) (or (2')) for a linear differential operator of order k .

In conclusion we give an example of an application of the results obtained to the study of noncoercive boundary value problems for elliptic equations.

Let Ω be a compact n -dimensional Riemannian manifold with smooth boundary Γ , and let ν be a nondegenerate vector field on Γ . We consider the problem of finding a function $u(x)$ for which

$$\Delta u = f \quad \text{in } \Omega; \quad \frac{\partial u}{\partial \nu} = g \quad \text{on } \Gamma.$$

Let Γ_0 be a smooth manifold in Γ of dimension $n-2$, and suppose that the field ν is not tangent to Γ at points of $\Gamma \setminus \Gamma_0$, while on Γ_0 it has contact with Γ of order k . This means that in a local coordinate system whose center coincides with some point on Γ_0 , and such that the domain Ω is described by the inequality $x_n > 0$, while Γ_0 is given by the equations $x_n = x_{n-1} = 0$, the boundary condition is written in the form

$$x_{n-1}^k \frac{\partial u}{\partial x_n} + \sum_{j=1}^{n-1} \nu_j(x) \frac{\partial u}{\partial x_j} = g(x).$$

Such problems were studied in [3], but there the order of contact was not taken into account. For $k = 1$ this problem was considered in [2]. If x_n coincides with the length of the arc of the geodesic normal to Γ , then the function $v(x) = u(x_1, \dots, x_{n-1}, 0)$ satisfies the equation

$$Pv = g + Lf,$$

where the principal part of the symbol of the operator P is equal to

$$p^0(x, \xi) = x_1^k |\xi| + i \sum_{j=1}^{n-1} \nu_j(x) \xi_j, \quad (3)$$

and L is a linear operator $H^s(\Omega) \rightarrow H^{s+1/2}(\Gamma)$. We require that the field ν not be tangent to Γ_0 , i.e., that $\nu_{n-1}(x) \neq 0$. It is easy to verify that for the symbol

(3) the inequality (2') holds if k is odd and $\nu_{n-1}(x) < 0$ on Γ_0 , or if k is even and $\nu_{n-1}(x) \neq 0$. Hence it follows that, under the indicated conditions, the solution $u(x)$ of our problem belongs to the space $H^s(\Omega)$, provided only that $f \in H^{s-(k+2)/(k+1)}(\Omega)$ and $g \in H^{s-3/2+k/(k+1)}(\Gamma)$. The example,

given in [3], shows the impossibility of improving this result.

Remark 1. It can be shown that operators of principal type are hypoelliptic (see [4]).

Remark 2. Suppose decompositions of the bundles E and F into direct sums are given,

$$E = \bigoplus_{k=1}^K E_k, \quad F = \bigoplus_{j=1}^J F_j$$

and two sequences of real numbers $s = (s_1, \dots, s_K)$, $t = (t_1, \dots, t_J)$. We agree to say that a pseudodifferential operator P from E to F has type (s, t) if, for every compact set K ,

$$\|Pu\|_{(t)} \leq C\|u\|_{(s)}, \quad u \in C_0^\infty(K, E),$$

where

$$\|u\|_{(s)} = \left(\sum_{k=1}^K \|u_k\|_{s_k}^2 \right)^{1/2}.$$

The results of the present note are readily extended to such operators if estimate (1) is replaced by the inequality

$$\|u\|_{(s)} \leq C \left(\|Pu\|_{(t+\delta)} + \|u\|_{(r)} \right), \quad u \in C_0^\infty(K, E),$$

where $t + \delta = t_1 + \delta_1, \dots, t_J + \delta$ and $0 \leq \delta < 1$, and the principal part $p^0(x, \xi) = (p_{jk}^0)_{\substack{k=1, \dots, K \\ j=1, \dots, J}}$ of the symbol $p(x, \xi) = (p_{jk})_{\substack{k=1, \dots, K \\ j=1, \dots, J}}$ is taken to be those terms of the matrix $p(x, \xi)$ for which the order p_{jk}^0 is equal to $s_k - t_j$.

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