

# PERIODIC NILPOTENT LINEAR GROUPS OVER THE FIELD OF RATIONAL NUMBERS

R. T. Vol'vachev

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## Abstract

## Full Text

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MATHEMATICS

R. T. VOL' VACHEV

# PERIODIC NILPOTENT LINEAR GROUPS OVER THE FIELD OF RATIONAL NUMBERS

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The theory of linear nilpotent and locally nilpotent groups was developed by D. A. Suprunenko (<sup>1-6</sup>). He also studied periodic linear locally nilpotent groups over an algebraically closed field of characteristic zero (<sup>7</sup>).

In the present note we study periodic nilpotent subgroups of the full linear group  $GL(n, R)$  over the field of rational numbers  $R^*$ . It is proved that two maximal periodic nilpotent subgroups of  $GL(n, R)$  are conjugate in  $GL(n, R)$  if their centers are conjugate in  $GL(n, R)$ . It is also shown that in  $GL(n, R)$  there exists, up to conjugacy, only a finite number of maximal nilpotent periodic subgroups. A complete description is given of the maximal periodic nilpotent subgroups of  $GL(n, R)$ .

1. We first consider irreducible nilpotent periodic subgroups of  $GL(n, R)$ . Their structure is closely connected with the structure of linear  $p$ -groups over the field of rational numbers (see Theorem 1). (For results on linear  $p$ -groups see (<sup>9</sup>), where Sylow  $p$ -subgroups of the full linear group over an arbitrary field are studied.)

The structure of irreducible nilpotent periodic subgroups is described by the following theorem.

**Theorem 1.** Let  $\Gamma$  be an irreducible periodic nilpotent subgroup of  $GL(n, R)$ . Then  $\Gamma$  is conjugate in  $GL(n, R)$  to a certain subgroup of the group of all matrices  $g$  of the form

$$g = a_1 \times a_2 \times \cdots \times a_\nu, \quad (1)$$

where  $\times$  is the sign of the Kronecker product,  $a_i$  ranges over an irreducible  $p_i$ -subgroup of a Sylow subgroup of the group  $GL(\varphi(p_i^{\alpha_i}), R)$  ( $\varphi$  is Euler's function). Consequently,

$$n = \varphi(k), \quad (2)$$

where  $k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\nu^{\alpha_\nu}$  is the canonical decomposition of the number  $k$  (see (1)), and the group of all matrices of the form (1) is determined uniquely, up to conjugacy in  $GL(n, R)$ , by specifying the number  $k$  from (2).

**Corollary.** If  $n \neq \varphi(k)$ , where  $k$  is an integer and  $\varphi$  is Euler's function, then in  $GL(n, R)$  there are no irreducible nilpotent periodic subgroups.

Now the following question naturally arises: which groups of matrices of the form (1) are maximal irreducible nilpotent periodic subgroups of  $GL(n, R)$ , or, in other words, to which numbers  $k$  satisfying the condition  $n = \varphi(k)$  does there correspond a maximal irreducible nilpotent periodic subgroup of  $GL(n, R)$ .

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\* We note that every locally nilpotent subgroup of  $GL(n, R)$  is nilpotent <sup>(6,8)</sup>.

Let us turn to the solution of this question. It is easy to see that the group of matrices of the form (1) can be regarded as a subgroup of  $GL(r, \Sigma)$ , where

$$\Sigma = R(\varepsilon), \quad (3)$$

$\varepsilon$  is a primitive root of unity of degree  $m$ , and (cf. (2))

$$m = p_1 p_2 \dots p_\nu, \quad (4)$$

$$r = n/\varphi(m). \quad (5)$$

We shall first show that the group of all matrices of the form (1) is almost always maximal among irreducible nilpotent periodic subgroups whose linear  $R$ -envelope of the center coincides with  $\Sigma$ . More precisely, the following is true.

**Lemma 1.** *Let  $\Gamma$  be the group of all matrices of the form (1), determined by such a number  $k$  that  $n = \varphi(k)$ , and let  $k = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_\nu^{\alpha_\nu}$  be the canonical factorization of the number  $k$ , and let  $\Sigma = R(\varepsilon)$ , where  $\varepsilon$  is a primitive root of unity of degree  $m = p_1 p_2 \dots p_\nu$ . Then  $\Gamma$  is maximal among irreducible nilpotent periodic subgroups of the group  $GL(n, R)$  whose linear  $R$ -envelopes of the centers coincide with  $\Sigma$ , except for the case when  $\Gamma$  is determined by a number  $k$  satisfying the relation  $n = \varphi(k) = \varphi(2k)$ .*

Now one can answer the question formulated earlier. The answer is given by the following

**Theorem 2.** *For any  $k$  such that  $n = \varphi(k)$ , the group of all matrices of the form (1), determined by this number  $k$ , is a maximal irreducible nilpotent periodic subgroup of  $GL(n, R)$ , except for the case when the group (1) is determined by such a number  $k$  for which the relation  $n = \varphi(k) = \varphi(2k)$  is possible.*

From Theorems 1 and 2 there follows the following

**Theorem 3.** *Two maximal irreducible nilpotent periodic subgroups of  $GL(n, R)$  are conjugate in  $GL(n, R)$  if their centers are conjugate in  $GL(n, R)$ .*

The theorems formulated above give a complete classification and construction of maximal irreducible nilpotent periodic subgroups of  $GL(n, R)$ . For convenience, let us combine the results obtained as follows.

**First main theorem.** *If  $n \neq \varphi(k)$ , where  $\varphi$  is Euler's function and  $k$  is an integer, then in  $GL(n, R)$  there are no irreducible nilpotent periodic subgroups. Let  $\mathfrak{M}$  be the set of all such numbers  $k$  that  $n = \varphi(k)$ , except for that number  $k_1$  for which the relation  $n = \varphi(k_1) = \varphi(2k_1)$  is possible, and let*

$$k = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}, \quad (6)$$

be the canonical factorization of the number  $k$ .

Then:

1. *For any  $k$  from  $\mathfrak{M}$ , in  $GL(n, R)$  there is a maximal irreducible nilpotent periodic subgroup whose matrices can be represented in the form*

$$g = a_1 \times a_2 \times \cdots \times a_\nu,$$

where  $a_i$  runs through a  $p_i$ -subgroup of the Sylow group  $GL(\varphi(p_i^{\alpha_i}), R)$  (see (6)), and  $\times$  denotes the Kronecker product.

2. *Different  $k$  from  $\mathfrak{M}$  determine nonisomorphic groups.*
3. *Every maximal irreducible nilpotent periodic subgroup of  $GL(n, R)$  is conjugate in  $GL(n, R)$  to the group determined by some  $k$  from  $\mathfrak{M}$ .*

**Corollary.** In  $GL(n, R)$ , up to conjugacy, there exists only a finite number of maximal irreducible nilpotent periodic subgroups.

- II. Let now  $\Gamma$  be a maximal nilpotent periodic subgroup of  $GL(n, R)$ . As is known, any periodic subgroup of  $GL(n, R)$  is completely reducible (<sup>13</sup>, <sup>14</sup>). Consequently,  $\Gamma$  is a completely reducible group.

Let  $G$  be one of the irreducible components of the group  $\Gamma$ . If  $m$  is the degree of  $G$ , then  $m = \varphi(k)$ , and  $G$  is a maximal irreducible nilpotent periodic subgroup of  $GL(m, R)$ . Consequently, a maximal periodic nilpotent subgroup of the group  $GL(n, R)$  is uniquely determined, up to conjugacy in  $GL(n, R)$ , by the degrees  $n_1, n_2, \dots, n_s$  of its irreducible components and by the centers of these irreducible groups, or, in other words, by a set of numbers  $k_1, k_2, \dots, k_s$  such that, for every  $k_i$ ,

$$\varphi(k_i) \neq \varphi(2k_i), \quad (7)$$

where the numbers  $n_i$ , where

$$n_i = \varphi(k_i), \quad (8)$$

satisfy the relation

$$n = n_1 + n_2 + \dots + n_s, \quad n_i > 0. \quad (9)$$

Thus, it is necessary to determine to what set of numbers  $k_1, k_2, \dots, k_s$ , satisfying (7), (8), and (9), there corresponds a maximal nilpotent periodic subgroup of  $GL(n, R)$ . We shall formulate this assertion more precisely.

Let  $n$  be represented in the form (9), with each  $k_i$  from (8) satisfying (7). Represent the space  $R^n$ , in which  $GL(n, R)$  acts, in the form

$$R^n = \Sigma_1 + \dots + \Sigma_s, \quad (10)$$

where  $\Sigma_i$  is a subspace of dimension  $n_i$ . By  $\Gamma_{k_i}$  we shall denote the subgroup of  $GL(n, R)$  which induces on each  $\Sigma_j$  ( $j \neq i$ ) the identity group, and on  $\Sigma_i$  the maximal irreducible nilpotent periodic group of degree  $n_i$  determined by the number  $k_i$ , with  $n_i = \varphi(k_i)$ .

Construct the group

$$\Gamma_{k_1, k_2, \dots, k_s} = \Gamma_{k_1} \Gamma_{k_2} \dots \Gamma_{k_s}. \quad (11)$$

It turns out that the group  $\Gamma_{k_1, k_2, \dots, k_s}$  is not always a maximal nilpotent periodic subgroup of  $GL(n, R)$ . Our task is to find conditions under which the sets of numbers  $k_1, k_2, \dots, k_s$ , satisfying (7), (8), and (9), determine a maximal nilpotent periodic subgroup  $\Gamma_{k_1, k_2, \dots, k_s}$  of the group  $GL(n, R)$ . The method used by us for this goes back to <sup>(15)</sup>.

**Lemma 2.** If the formula

$$i \neq j \Rightarrow k_i \neq k_j$$

is true, then  $\Gamma_{k_1, k_2, \dots, k_s}$  is a maximal nilpotent periodic subgroup of  $GL(n, R)$ .

**Lemma 3.** Let  $k_1 = k_2 = \dots = k_s = k$ . The group  $\Gamma_{k_1, k_2, \dots, k_s}$  is contained in some nilpotent periodic subgroup of the group  $GL(n, R)$  only when  $k = 2^\alpha$ ,  $s = 2^\beta$ .

From the last two lemmas it follows:

**Second main theorem.** A reducible nilpotent periodic group  $\Gamma_{k_1, \dots, k_s}$ , inducing in each irreducible block of degree  $n_i > 0$  a maximal irreducible nilpotent periodic group determined by the number  $k_i$ , where  $n_i = \varphi(k_i)$ , is not only then ...

is a maximal nilpotent periodic subgroup of  $GL(n, R)$ ,  $n = n_1 + \dots + n_s$ , when among  $k_1, k_2, \dots, k_s$  there are at least two such numbers  $k_{j_1}$  and  $k_{j_2}$  that  $k_{j_1} = k_{j_2} = 2^\alpha$ .

At the same time, the following assertions follow from this.

**Theorem 4.** In  $GL(n, R)$ , up to conjugacy in  $GL(n, R)$ , there exists only a finite number of maximal nilpotent periodic subgroups.

**Theorem 5.** Two maximal nilpotent periodic subgroups of  $GL(n, R)$  are conjugate in  $GL(n, R)$  if their centers are conjugate in  $GL(n, R)$ .

In conclusion, I express my deep gratitude to D. A. Suprunenko for discussion of a number of questions connected with the present note.

Belorussian State University  
named after V. I. Lenin

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*Note: Figure translations are in progress. See original paper for figures.*

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