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Abstract

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MATHEMATICS

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AN ANALOGUE OF STONE'S THEOREM IN A SPACE WITH AN INDEFINITE METRIC

(Presented by Academician L. S. Pontryagin on 18 I 1966)

1. Let $\mathfrak{H}$ be a Hilbert space; let P be an orthoprojector in $\mathfrak{H}$, different from 0 and 1; let $Q = 1 - P$, $J = P - Q$, $\varkappa = \min\{\dim P, \dim Q\}$. Put $[x, y] = (Jx, y)$, where (x, y) is the scalar product in $\mathfrak{H}$. The space $\mathfrak{H}$, with the indefinite form $[x, y]$ thus defined, is called a space $\Pi_\varkappa$.^{*} In a natural way one introduces the notions of an isometric, unitary, Hermitian, and self-adjoint operator with respect to $[x, y]$; they are called, respectively, a J -isometric, J -unitary, J -Hermitian, and J -self-adjoint operator in $\Pi_\varkappa$.

Let A be a J -Hermitian operator. If there exists a non-real number λ such that λ and $\bar{\lambda}$ belong to the resolvent set of A , then, as is not hard to show, A is a J -self-adjoint operator.

2. A family $U(t)$, $-\infty < t < \infty$, of operators in $\Pi_\varkappa$ is called a **one-parameter group of J -unitary operators in $\Pi_\varkappa$** if 1) $U(t)$ is a J -unitary operator for each $t \in (-\infty, \infty)$; 2) $U(t_1 + t_2) = U(t_1)U(t_2)$ for all $t_1, t_2 \in (-\infty, \infty)$; 3) $U(t)x$ is a strongly continuous (in the sense of the norm in $\Pi_\varkappa = \mathfrak{H}$) function of t on $(-\infty, \infty)$ for every $x \in \Pi_\varkappa$.

An analogue of the well-known Stone theorem is the following

Theorem 1. *The generating operator A of any one-parameter group of J -unitary operators in $\Pi_\varkappa$ has the form $A = iH$, where H is a J -self-adjoint operator in $\Pi_\varkappa$ such that, for all sufficiently large in modulus integers n ($|n| \geq n_0$) and some constants $M > 0$, $\beta > 0$: 1) $(1 - in^{-1}H)^{-1}$ exists and is bounded; 2)*

$$\|(1 - in^{-1}H)^{-m}\| \leq M(1 - |n^{-1}\beta|)^{-m}, \quad m = 1, 2, 3, \dots$$

Conversely, for every J -self-adjoint operator H in $\Pi_\varkappa$ satisfying conditions 1) and 2), the operator $A = iH$ is the generating operator of a one-parameter group of J -unitary operators $U(t)$ in $\Pi_\varkappa$. The correspondence $U(t) \leftrightarrow H$, thus established, is one-to-one.

Proof. Let $U(t)$ be a one-parameter group of J -unitary operators in $\Pi_\varkappa$. Considering $\Pi_\varkappa$ as a Banach space with the usual Hilbert norm in $\mathfrak{H}$, we conclude that (see, for example, ⁽⁴⁾, Ch. IX, Secs. 1, 2 and 9): a) $\|U(t)\| \leq Me^{\beta|t|}$ for

some constants $M > 0, \beta > 0$; b) the generating operator A of the one-parameter group $U(t)$ satisfies the conditions: b₁) the domain of definition $D(A)$ of the operator A is dense in Π_{κ} ; b₂) $(1 - n^{-1}A)^{-1}$ exists and is bounded and

$$\|(1 - n^{-1}A)^{-m}\| \leq M(1 - |n^{-1}\beta|)^{-m}, \quad m = 1, 2, \dots,$$

for all sufficiently large in modulus integers n .

Put $A = iH$ and prove that H is a J -self-adjoint operator in Π_{κ} . Since $U(t)$ is a J -unitary operator, we have $[U(t)x, U(t)y] = [x, y]$ for all $x, y \in \Pi_{\kappa}$. Taking here $x, y \in D(H) = D(A)$ and differentiating with respect to t at the point $t = 0$, we obtain

$$[Ax, y] + [x, Ay] = 0, \quad \text{i.e. } i[Hx, y] -$$

* For $\kappa < \infty$ the basic properties of the space Π_{κ} and of operators in it are considered in ^(1, 2); for more general spaces with indefinite metric see ⁽³⁾.

$-i[x, Hy] = 0$ for all $x, y \in D(H)$. This means that H is J -Hermitian. Setting $A = iH$ in b₂), we conclude that H satisfies conditions 1) and 2). But then, by the remark at the end of Sec. 1, H is a J -self-adjoint operator in Π_{κ} . Conversely, let H be a J -self-adjoint operator in Π_{κ} satisfying conditions 1) and 2). Put $A = iH$. Then the operator A satisfies conditions b₁) and b₂), and therefore (see, for example, (4), Ch. IX, Sec. 9) A is the generating operator of a strongly continuous group of bounded operators $U(t)$ in Π_{κ} . We shall prove that $U(t)$ is J -unitary. Since $\frac{d}{dt}U(t)x = U(t)Ax = AU(t)x$ for all $x \in D(A)$, we have

$$\frac{d}{dt}[U(t)x, U(t)y] = [AU(t)x, U(t)y] + [U(t)x, Au(t)y] = i\{[HU(t)x, U(t)y] - [U(t)x, HU(t)y]\} = 0;$$

hence

$$[U(t)x, U(t)y] = \text{const} = [U(0)x, U(0)y] = [x, y].$$

Consequently, $U(t)$ is J -isometric; its J -unitarity now follows from the equality

$$U(t)U(-t) = U(-t)U(t) = U(0) = 1.$$

Finally, the correspondence $U(t) \rightarrow H$ is one-to-one, since the correspondence $U(t) \rightarrow A$ is one-to-one.

If iH is the generating operator of the group $U(t)$, we shall write $U(t) = \exp(itH)$. This notation is justified by the fact that, for some set D dense in Π_{κ} ,

$$U(t)x = x + \frac{1}{1!}itHx + \frac{1}{2!}(itH)^2x + \dots \quad \text{for } x \in D,$$

where the series on the right-hand side converges absolutely (see, for example, (5)).

Remark 1. If $A = iH$ is the generating operator of a one-parameter group of J -unitary operators in Π_{κ} , then conditions 1) and 2) of Theorem 1 are also fulfilled for all nonzero n for which $|\operatorname{Im} n|$ is greater than some constant.

Remark 2. If $H = H_1 + iH_2$, where H_1, H_2 are ordinary self-adjoint operators in $\mathfrak{H} = \Pi_{\kappa}$, and H_2 is bounded, while H is a J -self-adjoint operator in Π_{κ} , then H satisfies conditions 1) and 2) of Theorem 1.

Indeed,

$$\begin{aligned} 1 - in^{-1}H &= 1 - in^{-1}H_1 + n^{-1}H_2 = \\ &= (1 - in^{-1}H_1) [1 + (1 - in^{-1}H_1)^{-1}n^{-1}H_2], \end{aligned} \quad (1)$$

where $(1 - in^{-1}H_1)^{-1}$ exists and $\|(1 - in^{-1}H_1)^{-1}\| \leq 1$. Therefore

$$\|(1 - in^{-1}H_1)^{-1}n^{-1}H_2\| \leq |n|^{-1}\|H_2\| < 1 \quad \text{when } |n| > \|H_2\|. \quad (2)$$

Consequently, for $|n| > \|H_2\|$ there exists $(1 + (1 - in^{-1}H_1)^{-1}n^{-1}H_2)^{-1}$, and

$$\|(1 + (1 - in^{-1}H_1)^{-1}n^{-1}H_2)^{-1}\| \leq (1 - |n|^{-1}\|H_2\|)^{-1}.$$

But then from (1) and (2) we conclude that for $|n| > \|H_2\|$ there also exists

$$(1 - in^{-1}H)^{-1} = [1 + (1 - in^{-1}H_1)^{-1}n^{-1}H_2]^{-1}(1 - in^{-1}H_2)^{-1}$$

and

$$\|(1 - in^{-1}H)^{-m}\| \leq (1 - |n|^{-1}\|H_2\|)^{-m}, \quad m = 1, 2, \dots$$

From Remark 2 and Theorem 1 there follows

Theorem 2. In the space Π_{κ} , $\kappa < \infty$, every one-parameter group of J -unitary operators $U(t)$ is represented in the form $U(t) = \exp(itH)$, where H is a J -self-adjoint operator; conversely, for every J -self-adjoint H there exists such a one-parameter group of J -unitary operators $U(t)$ that $U(t) = \exp(itH)$.

Indeed, every J -self-adjoint operator in Π_{∞} , $\infty < \infty$, can be represented in the form $H = H_1 + iH_2$, where H_1, H_2 are ordinary self-adjoint operators and H_2 is bounded.

3. Theorem 1 carries over, with the appropriate changes, to strongly continuous one-parameter semigroups of J -isometric bounded operators $U(t)$, $t \geq 0$. In this case, in conditions 1 and 2 of Theorem 1, n must be regarded as positive, and H as a maximal J -Hermitian operator*.

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* *Note added in proof.* As the author has learned, Theorem 2 was formulated by M. G. Krein in his lecture at the IV All-Union Mathematical Congress (6) and proved by Shah Tao-shing (7).

Note: Figure translations are in progress. See original paper for figures.

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