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Abstract

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MATHEMATICS

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ON THE APPROXIMATION OF FUNCTIONS BY FOURIER-JACOBI SUMS

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Let $\{\hat{P}_k^{(\alpha,\beta)}(x)\}_0^\infty$ be a system of Jacobi polynomials orthonormal on the interval $-1 \leq x \leq 1$ with weight $n(x) = (1-x)^\alpha(1+x)^\beta$ ($\alpha, \beta > -1$), and let $R_n^{(\alpha,\beta)}(f, x) = f(x) - S_n^{(\alpha,\beta)}(f, x)$ be the n -th remainder of the Fourier-Jacobi series of a function $f(x)$ belonging to the class $W^{(r)}H^{(\gamma)}$ of all functions whose r -th derivative ($r \geq 0$) on the interval $-1 \leq x \leq 1$ satisfies a Lipschitz condition of order γ ($0 < \gamma \leq 1$) with constant equal to one.

In the present note, for α and $\beta \geq 0$, an estimate is obtained for the quantity

$$\sup_{f \in W^{(r)}H^{(\gamma)}} \|R_n^{(\alpha,\beta)}(f, x)\|_C,$$

where $\|u(x)\|_C = \max_{-1 \leq x \leq 1} |u(x)|$. In addition, for the case of ultraspherical polynomials ($\alpha = \beta = \lambda - \frac{1}{2}$) it is proved that the estimate obtained is exact in the sense of order.

In what follows $q = \max(\alpha, \beta)$, and all constants $C_1, \dots$ depend neither on x , nor on n , nor on the function $f \in W^{(r)}H^{(\gamma)}$.

Theorem 1. Let $r \geq 0$, $0 < \gamma \leq 1$, $r + \gamma > \frac{1}{2}$. If $\alpha, \beta > 0$, $q \geq \frac{1}{2}$, then the estimate

$$\sup_{f \in W^{(r)}H^{(\gamma)}} \|R_n^{(\alpha,\beta)}(f, x)\|_C \leq \frac{C_1}{(n+1)^{r+\gamma-q-\frac{1}{2}}}, \quad (1)$$

is valid, while if $\alpha, \beta > 0$, $q < \frac{1}{2}$, or if $\alpha\beta = 0$ ($\alpha \geq 0$, $\beta \geq 0$), then

$$\sup_{f \in W^{(r)}H^{(\gamma)}} \|R_n^{(\alpha,\beta)}(f, x)\|_C \leq \frac{C_2 \ln(n+2)}{(n+1)^{r+\gamma-q-\frac{1}{2}}}. \quad (2)$$

Theorem 2. Let $a > -\frac{1}{2}$, $r \geq 0$, $0 < \gamma \leq 1$. Then there exists a constant $C_3 > 0$ such that

$$\sup_{f \in W^{(r)}H^{(\gamma)}} |R_n^{(\alpha, \alpha)}(f, 1)| \geq \frac{C_3}{(n+1)^{r+\gamma-\alpha-\frac{1}{2}}}. \quad (3)$$

The proof of Theorem 1 is carried out with the aid of the strengthened Jackson theorem (see ⁽¹⁾, p. 276), by virtue of which, for any $n > r$, for every function $f(x) \in W^{(r)}H^{(\gamma)}$ one can construct an algebraic polynomial $Q_n(x)$ of degree $\leq n$ such that

$$|f(x) - Q_n(x)| \leq \frac{C_4}{n^{r+\gamma}} \left[(\sqrt{1-x^2})^{r+\gamma} + \frac{1}{n^{r+\gamma}} \right] \quad (-1 \leq x \leq 1).$$

It follows from this that

$$\begin{aligned} |R_n^{(\alpha, \beta)}(f, x)| &\leq |f(x) - Q_n(x)| + \int_{-1}^1 |f(t) - Q_n(t)| |K_n^{(\alpha, \beta)}(x, t)| n(t) dt \leq \\ &\leq \frac{C_5}{(n+1)^{r+\gamma}} + \frac{C_6}{(n+1)^{r+\gamma}} \int_{-1}^1 (1-t)^{r/2+\gamma/2+\alpha} (1+t)^{r/2+\gamma/2+\beta} |K_n^{(\alpha, \beta)}(x, t)| dt + \\ &\quad + \frac{C_7}{(n+1)^{2(r+\gamma)}} \int_{-1}^1 |K_n^{(\alpha, \beta)}(x, t)| n(t) dt = \\ &= \frac{C_5}{(n+1)^{r+\gamma}} + \frac{C_6}{(n+1)^{r+\gamma}} I_1 + \frac{C_7}{(n+1)^{2(r+\gamma)}} I_2, \end{aligned} \quad (4)$$

where

$$K_n^{(\alpha, \beta)}(x, t) = \sum_{k=0}^n \hat{P}_k^{(\alpha, \beta)}(x) \hat{P}_k^{(\alpha, \beta)}(t).$$

For the integral I_2 we have

$$\begin{aligned} I_2 &\leq \left\{ \int_{-1}^1 |K_n^{(\alpha, \beta)}(x, t)|^2 n(t) dt \right\}^{1/2} \left\{ \int_{-1}^1 n(t) dt \right\}^{1/2} = \\ &= \left\{ \sum_{k=0}^n |\hat{P}_k^{(\alpha, \beta)}(x)|^2 \right\}^{1/2} \left\{ \int_{-1}^1 n(t) dt \right\}^{1/2} \leq C_8 \left\{ \sum_{k=0}^n (k+1)^{2q+1} \right\}^{1/2} \leq C_9 (n+1)^{q+1}. \end{aligned} \quad (5)$$

From estimate (5) it follows that the last term in (4) is $O((n+1)^{q+1-2r-2\gamma})$ uniformly in x and f .

The estimate of the integral I_1 in the case $\alpha, \beta > 0$, $q \geq 1/2$ is carried out with the help of the formula

$$P_{n-1}^{(\alpha, \beta)}(x) = \frac{1}{(1-x)^{\alpha/2+1}(1+x)^{\beta/2+1}} \times \\ \times \left\{ \int_{-1}^x (1-t)^{\alpha/2}(1+t)^{\beta/2} \left[\frac{\alpha}{2} - \frac{\beta}{2} + \left(\frac{\alpha}{2} + \frac{\beta}{2} - 2 \right) t \right] P_{n-1}^{(\alpha, \beta)}(t) dt \right. \\ \left. - 2n \int_{-1}^x (1-t)^{\alpha/2}(1+t)^{\beta/2} P_n^{(\alpha-1, \beta-1)}(t) dt \right\} \quad (\alpha, \beta > 0), \quad (6)$$

which follows from the differential equation for Jacobi polynomials. From this formula it follows that for arbitrary α and $\beta > 0$ for which $q \geq 1/2$, and for $-1 < t < 1$, the inequality

$$\left| \frac{\hat{P}_n^{(\alpha, \beta)}(x) - \hat{P}_n^{(\alpha, \beta)}(t)}{x - t} \right| \leq \frac{C_{10}(n+1)^{q+1/2}}{(1-t)^{\alpha/2+1}(1+t)^{\beta/2+1}} \quad (-1 \leq x \leq 1), \quad (7)$$

holds, where the constant C_{10} depends neither on x , nor on t , nor on n . Using the Christoffel-Darboux formula ((2), p. 83) and inequality (7), for α and $\beta > 0$, $q \geq 1/2$ we obtain

$$I_1 \leq C_{11}(n+1)^{q+1/2} \times \\ \times \int_{-1}^1 (1-t)^{(r+\gamma+\alpha-2)/2}(1+t)^{(r+\gamma+\beta-2)/2} \left[|\hat{P}_n^{(\alpha, \beta)}(t)| + |\hat{P}_{n+1}^{(\alpha, \beta)}(t)| \right] dt. \quad (8)$$

But the integral in inequality (8) is bounded as $n \rightarrow \infty$ (see (2), p. 180). From inequalities (4), (5), and (8) the validity of estimate (1) follows.

In the cases where $\alpha, \beta > 0$, $q < 1/2$, or where $\alpha\beta = 0$ ($\alpha \geq 0$, $\beta \geq 0$), the integral I_1 is estimated with the help of a device applied by P. K. Suetin (5) to Legendre polynomials ($\alpha = \beta = 0$), and of an inequality proved by S. N. Bernstein (3):

$$|\hat{P}_n^{(\alpha, \beta)}(x)| (1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4} \leq C_{12} \quad (-1 \leq x \leq 1).$$

In proving Theorem 2 it is enough to restrict oneself to the consideration of functions $f(x) \in W^{(r)}H^{(\gamma)}$, for which $f(1) = f'(1) = \dots = f^{(r)}(1) = 0$.

We have, for any $\lambda > 0$ and $0 < \theta < \pi$,

$$(\sin \theta)^{2\lambda-1} P_n^{(\lambda)}(\cos \theta) = \frac{2^{2-2\lambda}}{\Gamma(\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1)} \sum_{\nu=0}^{\infty} f_{\nu n}^{(\lambda)} \sin(n+2\nu+1)\theta, \quad (9)$$

$$f_{0n}^{(\lambda)} = 1; \quad f_{\nu n}^{(\lambda)} = \frac{(1-\lambda)(2-\lambda)\cdots(\nu-\lambda)}{\nu!} \times \\ \times \frac{(n+1)(n+2)\cdots(n+\nu)}{(n+\lambda+1)(n+\lambda+2)\cdots(n+\lambda+\nu)}, \quad \nu = 1, 2, \dots$$

In the case of nonintegral λ , this formula is given by G. Szegő⁽²⁾. Using the uniform convergence in λ of the series in (9), one can prove formula (9) for natural $\lambda = k$, with $f_{\nu n}^{(k)} = 0$ if $\nu \geq k$. With the aid of representation (9), the expression for the remainder $R_n^{(\lambda-1/2, \lambda-1/2)}(f, 1) = R_n^{(\lambda)}$, for arbitrary $\lambda > 0$, is brought to the form

$$R_n^{(\lambda)} = a_n^{(\lambda)} \left\{ \int_0^{2\pi} f(\cos \psi) \sum_{\nu=0}^{\infty} f_{\nu n}^{(\lambda)} [\cos(n+2\nu+1)\psi + \right. \\ \left. + \left(1 - \frac{\lambda}{n+\nu+1+\lambda}\right) \cos(n+2\nu+2)\psi \right] d\psi - \\ \left. - \int_0^{2\pi} f(\cos \psi) \sum_{\nu=0}^{\infty} f_{\nu n}^{(\lambda)} \frac{2\lambda}{n+\nu+1+\lambda} \frac{\sin(n+2\nu+\frac{3}{2})\psi}{2 \sin \psi/2} d\psi \right\}, \quad (10)$$

where

$$a_n^{(\lambda)} = \Gamma(n+2\lambda+1) [2^{2\lambda} \sqrt{\pi} \Gamma(\lambda + \frac{1}{2}) \Gamma(n+\lambda+1)]^{-1}.$$

The application of r -fold integration by parts and the Abel transformation in the first integral in (10), as well as the application of Jackson's estimate⁽⁴⁾ for the n -th remainder of the Fourier series of a 2π -periodic differentiable function in estimating the second integral in (10), makes it possible to bring expression (10) to the form

$$R_n^{(\lambda)} = a_n^{(\lambda)} \left\{ \frac{(-1)^r 2^{\lambda+1}}{(n+1)^r} \int_0^{\pi} \cos \left[\left(n + \frac{1}{2} + \lambda\right) \psi + \frac{(r+1-\lambda)\pi}{2} \right] \times \right. \\ \left. \times \cos \frac{\psi}{2} \sin^{r+\lambda-1} \psi f^{(r)}(\cos \psi) d\psi + I_3 \right\}, \quad (11)$$

where

$$I_3 = O\left(\frac{\ln(n+2)}{(n+1)^{r+\gamma+1}}\right), \quad \text{if } \lambda \geq 1;$$

$$I_3 = O\left(\frac{\ln(n+2)}{(n+1)^{r+\gamma+\varepsilon}}\right), \quad \text{if } 0 < \varepsilon < \lambda < 1.$$

From (11) and from the fact that

$$\sup_{\varphi(x) \in H^{(\gamma)}} \left| \int_0^\pi \cos \left[\left(n + \frac{1}{2} + \lambda \right) \psi + \frac{(r+1-\lambda)\pi}{2} \right] \cos \frac{\psi}{2} \sin^{r+\lambda-1} \psi \varphi(\cos \psi) d\psi \right|$$

has order $n^{-\gamma}$, the validity of Theorem 2 follows. The indicated order is attained on the function $v_n(x) \in H^{(\gamma)}$ ($-1 \leq x \leq 1$), defined by the formula

$$v_n(x) = \frac{1}{2} u_n(x) \operatorname{sign} \left\{ \cos \left[\left(n + \frac{1}{2} + \lambda \right) \arccos x + \frac{(r+1-\lambda)\pi}{2} \right] \right\},$$

where $u_n(x) = (x-a)^\gamma$ for $x \in [a, (a+b)/2]$ and $u_n(x) = (b-x)^\gamma$ for $x \in [(a+b)/2, b]$, while $[a, b]$ is any one of the parts into which the segment

$[-1, 1]$ is divided by the zeros of the function

$$\cos \left[\left(n + \frac{1}{2} + \lambda \right) \arccos x + \frac{(r+1-\lambda)\pi}{2} \right].$$

Let us note that in the case when $\alpha = \beta$ is an integer, inequality (1) follows from the estimate of the Lebesgue function of the Fourier-Jacobi series obtained by I. K. Daugavet ⁶.

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⁵ P. K. Suetin, DAN, 158, No. 6, 1275 (1964).

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Note: Figure translations are in progress. See original paper for figures.

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