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1966

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Abstract

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UDC 517.925:517.948.35:513.88

MATHEMATICS

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A CRITERION FOR THE DISCRETENESS OF THE SPECTRUM OF A SELF-ADJOINT SYSTEM OF FIRST-ORDER DIFFERENTIAL EQUATIONS WITH SLOWLY VARYING COEFFICIENTS

(Presented by Academician I. M. Vinogradov on VII 24, 1965)

We consider the self-adjoint operator in $L_2(-\infty, +\infty)$

$$D_0 y = i\Lambda_0 y' + Q_0(x)y(x) = i \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} y' + \begin{pmatrix} p & q \\ \bar{q} & r \end{pmatrix} y, \quad (1)$$

where the real numbers $\lambda_1 \cdot \lambda_2 \neq 0$, and the matrix-function $Q_0(x)$ is continuous everywhere on the axis and Hermitian in the unitary space E_2 . If $\lambda_1 = -\lambda_2 = 1$ and $p(x) \equiv r(x)$, then, in essence, we are dealing with the operator

$$T_0 y = I_0 y' + A_0(x)y(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y' + \begin{pmatrix} a & b \\ b & c \end{pmatrix} y \quad (2)$$

in the Euclidean space R_2 , since $T_0 = UD_0U^{-1}$ for $U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}$. The operator T_0 was studied in a number of papers by E. C. Titchmarsh in connection with the relativistic Dirac equation.

In note ⁽²⁾ certain sufficient conditions were proposed for discreteness of the spectrum of the operator D_0 . The main content of the present note is the proof of a necessary and sufficient condition for discreteness of the spectrum, valid in the class of operators with slowly varying coefficients.

Theorem. 1) *Let the matrix $Q_0(x)$ vary slowly, i.e.*

$$\|Q_0(x) - Q_0(t)\| \leq \text{const} \cdot \quad (3)$$

for all $|x - t| \leq 1$. Then the spectrum of the operator is discrete if and only if

$$\lim_{|x| \rightarrow \infty} \left[|q(x)|\sqrt{-4\lambda_1\lambda_2} - |\lambda_2 p(x) - \lambda_1 r(x)| \right] = +\infty. \quad (4)$$

2) If the matrix $Q_0(x)$ varies slowly, then the continuous spectrum of the operator D_0 is bounded above (it may also be empty) if and only if

$$\lim_{\lambda \rightarrow +\infty} \lim_{|x| \rightarrow \infty} \left[|q(x)|\sqrt{-4\lambda_1\lambda_2} - |(\lambda_2 p - \lambda_1 r) + \lambda(\lambda_1 - \lambda_2)| \right] = +\infty. \quad (5)$$

3) If the matrix $Q_0(x)$ varies slowly, then the continuous spectrum of the operator D_0 is bounded below (it may also be empty) if and only if

$$\lim_{\lambda \rightarrow -\infty} \lim_{|x| \rightarrow \infty} \left[|q(x)|\sqrt{-4\lambda_1\lambda_2} - |(\lambda_2 p - \lambda_1 r) + \lambda(\lambda_1 - \lambda_2)| \right] = +\infty. \quad (6)$$

Corollary. If $\lambda_1 = -\lambda_2 = 1$ and $p(x) \equiv r(x)$, then conditions (4), (5), (6) admit a reformulation in terms of the eigenvalues of the matrix $Q_0(x)$ (and hence also of the matrix $A_0(x)$):

* $\|B\|$, $\mu[B]$, and $\nu[B]$ will denote, respectively, the Euclidean norm, the smallest and the largest eigenvalues of

1') the spectrum is discrete if and only if

$$\lim_{|x| \rightarrow \infty} \nu[A_0(x)] = +\infty, \quad \lim_{|x| \rightarrow \infty} \mu[A_0(x)] = -\infty; \quad (7)$$

2') the continuous spectrum is bounded above if and only if

$$\lim_{|x| \rightarrow \infty} \nu[A_0(x)] = +\infty \quad \text{and the function } \mu[A_0(x)] \text{ is bounded above;} \quad (8)$$

3') the continuous spectrum is bounded below if and only if

$$\lim_{|x| \rightarrow \infty} \mu[A_0(x)] = -\infty \quad \text{and the function } \nu[A_0(x)] \text{ is bounded below.} \quad (9)$$

Thus, in this case the spectral alternative is valid (cf. ⁽¹⁾, p. 174): the continuous part of the spectrum of the operator D_0 either is absent, or else is unbounded at least on one side.

Proceeding to the proof of the theorem, consider, as in ⁽²⁾, the more general operator

$$Dy = i\Lambda y' + Q(x)y(x), \quad (10)$$

where the principal part contains a constant real diagonal invertible matrix Λ , and the matrix function $Q(x)$ is continuous and Hermitian in the unitary E_k ($k \geq 2$).

Lemma 1 (splitting principle; cf. ⁽¹⁾, §§ 1, 2). *The distance from a real point λ to the continuous spectrum of the operator (10) is equal to*

$$\lim_{r \rightarrow +\infty} \left(\inf_{y \in K(r, \infty)} \frac{\|Dy - \lambda y\|}{\|y\|} \right), \quad (11)$$

where $K(r, \infty)$ denotes the class of finite piecewise-smooth vector functions, the compact support of each of which is situated outside the interval $(-r, +r)$.

Corollary 1. *In order that the spectrum of the operator D be discrete, it is necessary and sufficient that*

$$\lim_{r \rightarrow +\infty} \left(\inf_{y \in K(r, \infty)} \frac{\|Dy\|}{\|y\|} \right) = +\infty. \quad (12)$$

Corollary 2. *In order that the continuous part of the spectrum of the operator D be bounded above, it is necessary and sufficient that*

$$\lim_{\lambda \rightarrow +\infty} \lim_{r \rightarrow +\infty} \left(\inf_{y \in K(r, \infty)} \frac{\|Dy - \lambda y\|}{\|y\|} \right) = +\infty. \quad (13)$$

The condition for boundedness of the continuous spectrum below and for its two-sided boundedness is equivalent, respectively, to the requirements that the expression (11) tend to infinity as $\lambda \rightarrow -\infty$ and as $|\lambda| \rightarrow \infty$.

Lemma 2 (splitting principle; cf. ⁽¹⁾, p. 66). *Put*

$$d(\lambda; \Delta) = d(\lambda; a, b) = \inf_{y \in N(\Delta)} \frac{\|Dy - \lambda y\|^2}{\|y\|^2},$$

where the class $N(\Delta) = N(a, b)$ consists of finite piecewise-smooth vector functions vanishing outside the interval $\Delta = [a, b]$. For any number h ($0 < h < b - a \leq \infty$) there exists in Δ an interval Δ' of length h such that

$$d(\lambda; \Delta) \leq d(\lambda; \Delta') \leq d(\lambda; \Delta) + \frac{32}{h^2} \|\Lambda\|^2. \quad (14)$$

Corollary 1'. The spectrum of the operator D is discrete if and only if

$$\lim_{|x| \rightarrow \infty} d(0; x, x+1) = +\infty. \quad (15)$$

Corollary 2'. The continuous part of the spectrum of the operator D is bounded above if and only if

$$\lim_{\lambda \rightarrow +\infty} \lim_{|x| \rightarrow \infty} d(\lambda; x, x+1) = +\infty. \quad (16)$$

The conditions for boundedness of the continuous spectrum below and for its two-sided boundedness coincide with (16) for $\lambda \rightarrow -\infty$ and $|\lambda| \rightarrow \infty$, respectively.

Suppose now that the matrix Q varies slowly. Then $d(0; \alpha, \alpha+1) = m(\alpha) + O(1)$, where

$$m(\alpha) = \inf_{y \in N(\alpha, \alpha+1)} \left[\int_{\alpha}^{\alpha+1} |i\Lambda y'(t) + Q(\alpha)y(t)|^2 dt \Big/ \int_{\alpha}^{\alpha+1} |y(t)|^2 dt \right].$$

Consequently, the spectrum is discrete if $\lim_{|\alpha| \rightarrow \infty} m(\alpha) = +\infty$. But, on the other hand, by virtue of inequality (14), $m(\alpha) = n(\alpha) + O(1)$, where

$$n(\alpha) = \inf_{y \in N(-\infty, +\infty)} \left[\int_{-\infty}^{+\infty} |i\Lambda y'(t) + Q(\alpha)y(t)|^2 dt \Big/ \int_{-\infty}^{+\infty} |y(t)|^2 dt \right].$$

Thus the spectrum is discrete if and only if $\lim_{|\alpha| \rightarrow \infty} n(\alpha) = +\infty$. To compute $n(\alpha)$, we apply the Fourier transform:

$$\begin{aligned} n(\alpha) &= \inf_z \left[\int_{-\infty}^{+\infty} |Q(\alpha)z(t) - t\Lambda z(t)|^2 dt \Big/ \int_{-\infty}^{+\infty} |z(t)|^2 dt \right] \\ &= \inf_{t \in (-\infty, +\infty)} \mu[(Q(\alpha) - t\Lambda)^2] \\ &= \inf_t \min_{|\xi|=1} [(\Lambda^2 \xi, \xi)t^2 - 2(\operatorname{Re} \Lambda Q \xi, \xi)t + (Q^2(\alpha)\xi, \xi)]. \end{aligned}$$

Thus we finally obtain: the spectrum of the operator D is discrete if and only if

$$\lim_{|\alpha| \rightarrow \infty} \left[\min_{|\xi|=1} X(\alpha; \xi) \right] = +\infty, \quad (17)$$

where

$$X(\alpha; \xi) = (\Lambda^2 \xi, \xi)(Q^2(\alpha)\xi, \xi) - [\operatorname{Re}(\Lambda Q \xi, \xi)]^2.$$

It is not difficult to show that for $k=2$ condition (17) is equivalent to condition (4) of the theorem.

Remark. The lemmas and condition (17) remain valid if in the principal part of the operator D there stands an arbitrary (not necessarily diagonal) constant Hermitian matrix Λ .

In conclusion, the authors express their gratitude to Prof. M. A. Naimark.

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Received
21 VII 1965

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2. V. V. Martynov, DAN, 165, No. 5 (1965).

Note: Figure translations are in progress. See original paper for figures.

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