

# ON REMAINDER TERMS

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1966

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**Abstract**

**Full Text**

UDC 519.21

**MATHEMATICS**

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## ON REMAINDER TERMS IN MULTIDIMENSIONAL LIMIT THEOREMS

*(Presented by Academician Yu. V. Linnik, 30 IX 1965)*

At the present time several estimates are known for the rate of convergence of the probability function  $P_n(A)$  of the normalized sum of independent identically distributed  $k$ -dimensional random vectors to the probability function  $\Phi(A)$  of the  $k$ -dimensional normal law. When the set  $A$  is a  $k$ -dimensional interval, the best estimate is due to G. Bergström <sup>(1)</sup>: if the summands have finite third moments, then

$$P_n(A) - \Phi(A) = O\left(\frac{1}{\sqrt{n}}\right).$$

When  $A$  belongs to the class of convex or Borel sets, it is known only that the remainder term has the form  $O(n^{-1/2} \ln^\alpha n)$ ,  $\alpha > 0$  (see <sup>(2,3)</sup>). The natural question arises: under the same conditions, is it possible to get rid of the factor  $\ln^\alpha n$ ? The first two theorems of the present paper are devoted to this question.

Consider a sequence of independent identically distributed  $k$ -dimensional random vectors  $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}, \dots, \vec{\xi}^{(n)}$  of Euclidean space  $R_k$ . Suppose that  $\vec{\xi}^{(1)}$  has finite moments of order  $s$  ( $s > 0$ ). Moreover, let  $M\vec{\xi}^{(1)} = 0$ .

We shall denote by  $(\mathbf{x}, \mathbf{y})$  the scalar product of vectors  $\mathbf{x}, \mathbf{y} \in R_k$ ; by  $\mathfrak{A}$  the class of all Borel sets in  $R_k$ ;  $P_n(\mathbf{m}) = P\{\vec{\xi}^{(1)} + \dots + \vec{\xi}^{(n)} = \mathbf{m}\}$ ;  $P_n(A)$  the probability function of the sum

$$S^{(n)} = \frac{1}{\sqrt{n}} \sum_{j=1}^n \vec{\xi}^{(j)};$$

$V$  the covariance matrix of the vector  $\vec{\xi}^{(1)}$ ;  $\lambda_1^{\nu_1} \lambda_2^{\nu_2} \dots \lambda_k^{\nu_k}$  the semi-invariants of order  $(\nu_1, \nu_2, \dots, \nu_k)$  of the random vector  $\vec{\xi}^{(1)}$ . Following R. Rao <sup>(2)</sup>, we define  $\tilde{P}_j(\vec{\omega})$  by means of the formal identity

$$\exp \left\{ \sum_{j=3}^{\infty} \frac{(\vec{\lambda}, \vec{\omega})^j}{j!} \left( \frac{1}{\sqrt{n}} \right)^{j-2} \right\} = \sum_{j=0}^{\infty} \left( \frac{1}{\sqrt{n}} \right)^j \tilde{P}_j(\vec{\omega}),$$

and  $P_j(-\varphi)(\mathbf{x})$  and  $P_j(-\Phi)(\mathbf{x})$  by the equalities

$$P_j(-\varphi)(\mathbf{x}) = \frac{1}{(2\pi)^k} \int_{\tilde{R}_k} \tilde{P}_j(i\mathbf{t}) e^{-i(\mathbf{t}, \mathbf{x}) - \frac{1}{2} \mathbf{t} V \mathbf{t}'} d\mathbf{t},$$

$$P_j(-\Phi)(\mathbf{x}) = \int_{\substack{y_1 \leq x_1 \\ \dots \\ y_k \leq x_k}} P_j(-\varphi)(\mathbf{y}) d\mathbf{y}.$$

**Theorem 1.** *If the distribution function of the vector  $\vec{\xi}^{(1)}$  has an absolutely continuous component, then*

$$\sup_{A \in \mathfrak{A}} \left| P_n(A) - \sum_{j=0}^{s-2} n^{-j/2} \int_A dP_j(-\Phi) \right| = o(n^{-(s-2)/2}).$$

**Theorem 2.** *If  $\vec{\xi}^{(1)}$  has the identity covariance matrix and takes values in the lattice  $L = \{\mathbf{m}\}$ ,  $\mathbf{m} = (m_1, m_2, \dots, m_k)$ ;  $m_1, m_2, \dots, m_k = 0, \pm 1, \dots$ , with maximal span of the distribution equal to 1, then for all  $k$ , uniformly in  $A \in \mathfrak{A}$ ,*

$$P_n(A) = \sum_{m/\sqrt{n} \in A} \sum_{j=0}^{s-2} n^{-(j+k)/2} P_j(-\varphi) \left( \frac{\mathbf{m}}{\sqrt{n}} \right) + o(n^{-(s-2)/2}). \quad (1)$$

In the case where  $\vec{\xi}^{(1)}$  has finite third moments, from (1), by means of the multidimensional Euler-Maclaurin summation formula, we obtain that for all  $k$ , uniformly in  $A \in \mathfrak{A}$ ,

$$P_n(A) = \int_A dQ(\mathbf{x}) + \frac{1}{\sqrt{n}} \int_A dP_1(-\Phi) + o\left(\frac{1}{\sqrt{n}}\right).$$

In particular, uniformly in  $\mathbf{x} \in R_k$ ,

$$F_n(\mathbf{x}) = \Phi(\mathbf{x}) + \frac{1}{\sqrt{n}} P_1(-\Phi)(\mathbf{x}) - \sum_{j=1}^k S(x_j \sqrt{n}) \frac{\partial \Phi(\mathbf{x})}{\partial x_j} + o\left(\frac{1}{\sqrt{n}}\right).$$

Here  $F_n(\mathbf{x})$  is the distribution function of the sum  $S^{(n)}$ ;  $\Phi(\mathbf{x})$  is the distribution function of the  $k$ -dimensional normal law with zero vector of mathematical expectations and identity covariance matrix,

$$Q(\mathbf{x}) = \prod_{j=1}^k \left[ 1 - \frac{1}{\sqrt{n}} S(x_j \sqrt{n}) \frac{\partial}{\partial x_j} \right] \Phi(\mathbf{x})$$

and  $S(u) = [u] - u + \frac{1}{2}$ , where  $[u]$  is the integer part of  $u$ .

Theorems 1 and 2 for all  $k < 2s$  are proved by the method of characteristic functions, and after this, with the help of the inequalities obtained, the proofs of the theorems for arbitrary  $k$  are completed by the method of truncation of random vectors.

**Theorem 3.** *Under the conditions of Theorem 2, for all  $k < 2s$ ,*

$$\sum_{m \in L} \left| P_n(\mathbf{m}) - \sum_{j=0}^{s-2} n^{-(j+k)/2} P_j(-\varphi) \left( \frac{\mathbf{m}}{\sqrt{n}} \right) \right| = o(n^{-(s-2)/2}). \quad (2)$$

Let us note that relation (2) with a remainder term  $O(n^{-(s-2)/2} \ln^{k/2} n)$  was obtained in (2).

**Theorem 4.** *Under the conditions of Theorem 2, for all  $k$ ,*

$$\sum_{m \in L} \left| \frac{\mathbf{m}}{\sqrt{n}} \right|^{2\nu} \left| P_n(\mathbf{m}) - \sum_{j=0}^{s-2} n^{-(j+k)/2} P_j(-\varphi) \left( \frac{\mathbf{m}}{\sqrt{n}} \right) \right|^2 = o(n^{-s-k/2+2})$$

for  $\nu = 0, 1, \dots, s$ .

Theorems 3 and 4 are proved by the method of characteristic functions.

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Received  
27 IX 1965

## REFERENCES

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*Note: Figure translations are in progress. See original paper for figures.*

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