

AN ANALOGUE OF THE RADÓ–CARTAN THEOREM FOR SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract

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MATHEMATICS

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AN ANALOGUE OF THE RADÓ–CARTAN THEOREM FOR SOLUTIONS OF ELLIPTIC EQUATIONS

(Presented by Academician M. A. Lavrent'ev on 10 VI 1965)

I. Formulation of the Results

The main result of this note is, in a certain sense, a generalization of the following theorem of Radó–Cartan ^{(1)*}.

A complex-valued function $f(z)$, $z = (x_1, x_2)$, continuous in a domain $G \subset R^2$, is analytic in the domain G if it is analytic in the open subset $G \setminus f^{-1}(0)$ ($f^{-1}(0) = \{z : f(z) = 0\}$) of the domain G .

Consider the linear elliptic equation of second order

$$Lu \equiv \sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^n a_i(x)u_{x_i} + a(x)u = f(x). \quad (1)$$

We shall assume that the coefficients of equation (1) are defined in a domain $G \subset R^n$ and that the condition

$$a_{ij} \in C'(G); \quad a_i, a, f \in C^\alpha(G), \quad \alpha > 0. \quad (2)$$

is satisfied. Let a function $u(x)$ also be given in the domain G . By $[u^{-1}(0)]$ we shall denote the subset of the domain G consisting of the boundary points of the set $u^{-1}(0)$ of zeros of the function $u(x)$. The analogue of the Radó–Cartan theorem for solutions of equation (1) is formulated as follows.

Theorem 1. *A function $u(x)$, continuously differentiable in the domain $G \subset R^n$, is a solution of the elliptic equation (1), satisfying conditions (2), if it is a solution of this equation in the open subset $G \setminus [u^{-1}(0)]$ of the domain G .*

The proof of Theorem 1 is given in Section III. In Section II an application of Theorem 1 to the question of uniqueness of the solution of the Cauchy problem is considered.

Remark 1. The assertion of Theorem 1 is obvious if the set $[u^{-1}(0)]$ is a surface of class C' . Indeed, a direct verification shows that in this case the function $u(x)$ satisfies equation (1) in the integral sense. It is known (see, for example, (3)) that every generalized solution of equation (1), under conditions (2), is classical. Thus, the regularity of the function $u(x)$ can fail only at those points $x \in [u^{-1}(0)]$ at which $\nabla u = (u_{x_1}, \dots, u_{x_n}) = 0$.

Remark 2. In the author's papers (4,5) ($n = 2$ and $n > 2$, respectively), the question of the "removability" of the zero set for solutions of equation (1) was considered for $f(x) \equiv 0$. It is easy to see that in this case the requirement that the equation be satisfied on the set $G \setminus [u^{-1}(0)]$ is equivalent to the requirement that the equation be satisfied on the set $G \setminus u^{-1}(0)$. It should be noted that the main result of (4) ($n = 2$) follows from the Radó–Cartan theorem formulated above and the theorem on the representation of solutions of a second-order elliptic system by analytic functions.

* In a paper of E. Heinz (2), a simpler proof is given, based, like our considerations, on the maximum principle.

We now formulate a generalization of Theorem 1 to the case of a nonlinear second-order equation arising in problems of the calculus of variations

$$\sum_{i=1}^n \frac{\partial}{\partial x_i} a_i[x, u(x), \nabla u(x)] + a[x, u(x), \nabla u(x)] = 0. \quad (3)$$

Theorem 1'. Let in a domain $G \subset R^n$ a function $u(x) \in C^1(G)$ and an equation of the form (3) be given. Suppose that the functions a_i, a have Hölder-continuous first derivatives with respect to all their arguments and that equation (3) is elliptic on the function $u(x)$.

Then, if the function $u(x)$ satisfies equation (3) in the open subset $G \setminus [u^{-1}(0)]$ of the domain G , it is a solution of this equation everywhere in the domain G .

We do not give the proof of Theorem 1'. Following the scheme of the proof of Theorem 1, it is not difficult to show that $u(x)$ is a generalized solution of the linear elliptic equation

$$\sum_{i,j=1}^n b_{ij}(x) u_{x_i x_j} = f(x), \quad b_{ij}(x) = \partial a_i(x, u, \nabla u) / \partial u_{x_j} \quad (3')$$

with continuous coefficients and right-hand side. In proving the second of relations (4) of Lemma 1, one must keep in mind that Remark 1 loses its force for equation (3') with nondifferentiable leading coefficients, and it should be used as applied to equation (3). After it has been proved that $u(x)$ is a generalized solution of equation (3'), and consequently also of equation (3), it follows from known results (see (3)) that $u(x) \in C^{2+\alpha}(G)$ and is a classical solution of equation (3).

II. On the Cauchy problem. Theorem 1 has an interesting application to the question of uniqueness of the solution of the Cauchy problem for equation (1) with data on a nonsmooth set γ . This application was pointed out to the author by M. M. Lavrent'ev.

Definition. We shall call a closed set $\gamma \subset R^n$ **separating the space R^n at a point $x_0 \in \gamma$** if, for every ball $B(x_0, a) = \{x : |x - x_0| < a\}$ of sufficiently small radius a , the open set $B(x_0, a) \setminus \gamma$ is not connected.

Theorem 2. The Cauchy problem for elliptic equation (1), satisfying conditions (2), with data on a set γ separating the space R^n at the point x_0 , has at most one solution. That is, there does not exist a function $u(x)$ of class C^1 in a neighborhood G of the point x_0 , different from zero, satisfying the following conditions:

$$Lu(x) = 0 \quad \text{for } x \in G \setminus \gamma; \quad u(x) = \nabla u(x) = 0 \quad \text{for } x \in G \cap \gamma.$$

Proof. Choose $a > 0$ so small that $B(x_0, a) \subset G$ and the set $B(x_0, a) \setminus \gamma$ is disconnected. Represent the latter set as the sum of two disjoint open sets B_1 and B_2 . Finally, define in the ball $B(x_0, a)$ the function $\tilde{u}(x)$:

$$\tilde{u}(x) = \begin{cases} u(x), & \text{for } x \in B_1, \\ 0, & \text{for } x \in B_2 \cup \gamma. \end{cases}$$

It is easy to see that the function $\tilde{u}(x)$ in $B(x_0, a)$ satisfies the conditions of Theorem 1. Consequently, $\tilde{u}(x) \in C^2(B(x_0, a))$ and $L\tilde{u} = 0$ everywhere in the ball $B(x_0, a)$. It is known (see, for example, ⁽⁶⁾) that equation (1), when conditions (2) are fulfilled, has the property of unique continuation*, i.e.

* In ⁽⁷⁾ an example is constructed of an elliptic equation (1) with leading coefficients $a_{ij}(x)$, Hölder-continuous

any solution of the homogeneous equation $Lu = 0$ that is equal to zero on an open set is identically equal to zero. Thus, $u = 0$ in B_1 , and from this, in turn, it follows that $u \equiv 0$ in G .

III. Proof of Theorem 1. We shall assume that equation (1) has the form

$$Lu \equiv \sum_{i,j=1}^n a_{ij}(x)u_{x_i x_j} + \sum_{i=1}^n a_i(x)u_{x_i} = f(x). \quad (1')$$

This entails no loss of generality, since, moving the term $a(x)u$ of equation (1) to the right-hand side, we see that the function $u(x)$ satisfies the conditions of Theorem 1 also for an equation of the form (1').

The proof of Theorem 1 has much in common with the author's work ⁽⁵⁾. The proof is based on two lemmas.

Lemma 1. *Let the function $u(x)$ satisfy the conditions of Theorem 1 with the operator L defined in (1'). Suppose that in the domain $G_1 \subset G$ a solution $v(x)$ of the equation $Lv = g > f$ is given and the maximum of the difference $v - u$ is attained at a point $x_0 \in G_1$. Then*

$$x_0 \in [u^{-1}(0)], \quad \nabla v(x_0) = \nabla u(x_0) = 0. \quad (4)$$

Proof of Lemma 1. Suppose that at the maximum point x_0 of the function $w = v - u$ at least one of the relations (4) is not fulfilled. Then, by Remark 1, the function $w(x) \in C^2$ in a neighborhood of the point x_0 , and $Lw(x_0) > 0$. On the other hand, it is known that for an elliptic operator L of the form (1'), at a maximum point the inequality $Lw(x_0) \leq 0$ must hold. The contradiction obtained proves Lemma 1.

Lemma 2. *Let the function $v(x)$ be analytic in the domain $G \subset R^n$. Then the image of any compact subset in G of the zero set of $\nabla v(x)$ under the mapping $x \rightarrow v(x)$ consists of a finite number of points.*

Lemma 2 (see ⁽⁵⁾) is an obvious consequence of the known property of local connectedness of the zero set of a real analytic function.

Let us consider a ball $B(x', \rho)$, $\overline{B}(x', \rho) \subset G$, and a function $\tilde{u}(x)$ which in this ball is the solution of the following Dirichlet problem for equation (1'):

$$L\tilde{u}(x) = f(x) \quad \text{for } x \in B(x', \rho); \quad \tilde{u}(x) = u(x) \quad \text{for } |x - x'| = \rho.$$

To prove the theorem it is enough to derive a contradiction from the assumption that $\tilde{u} \neq u$ in an arbitrarily chosen ball $B(x', \rho)$. Without loss of generality, we shall suppose that the difference $\tilde{u} - u$ takes positive values in the ball $B(x', \rho)$. We denote by $m > 0$ the maximum of the difference $\tilde{u} - u$ in the ball $B(x', \rho)$.

It is not difficult to show that there exists an analytic function $v(x)$ in the ball $B(x', \rho)$ satisfying the following conditions:

- 1) $Lv(x) > f(x)$ for $x \in B(x', \rho)$;
- 2) the maximum of the difference $v - u$ is attained at a point $x_0 \in B(x', \rho)$,
and

$$v(x_0) - u(x_0) > v(x) - u(x) \quad \text{for } |x - x'| = \rho.$$

As the function $v(x)$ one may take, for example, one of the terms of the sequence $v^{(k)}(x)$ (with sufficiently large index k), defined as follows:

$$\sum_{i,j=1}^n a_{ij}^{(k)}(x) v_{x_i x_j}^{(k)} + \sum_{i=1}^n a_i^{(k)}(x) v_{x_i}^{(k)} = f^{(k)}(x) \quad \text{for } x \in B(x', \rho),$$

$$v^{(k)}(x) = \varphi(x) \quad \text{for } |x - x'| = \rho.$$

Here $\varphi \in C^{2+\alpha}(\overline{B}(x', \rho))$, $|\varphi(x) - u(x)| < m/2$ for $|x - x'| = \rho$; $a_{ij}^{(k)}$, $a_i^{(k)}$, $f^{(k)}$ are analytic in $B(x', \rho)$ functions converging in the metric

of the space $C^\alpha(\overline{B}(x', \rho))$ to a_{ij} , a_i , and $f + h$, respectively; the quantity $h = \text{const} > 0$ is sufficiently small.

By Lemma 1, at the point x_0 of maximum of the difference $v - u$ the relations $u(x_0) = 0$, $\nabla v(x_0) = 0$ hold.

It is convenient to consider separately two cases:

- a) In some neighborhood of the point x_0 the inequality $v(x) \leq v(x_0)$ holds. Introduce the function $v_\varepsilon(x) = v(x) - \varepsilon|x - x_0|^2$, $\varepsilon > 0$.

It is easy to see that there exist $\varepsilon, \rho' > 0$ such that

$$Lv_\varepsilon(x) > f(x) \quad \text{for } x \in B(x_0, \rho');$$

$$\nabla v_\varepsilon(x) \neq 0 \quad \text{for } 0 < |x - x_0| < \rho'. \quad (5)$$

Since at the point x_0 the difference $v_\varepsilon(x) - u(x)$ has an isolated maximum, the function $v_\varepsilon(x + y) - u(x)$ (for sufficiently small $|y|$), considered in the ball $|x - x_0| < \rho'/2$, attains its maximum at an interior point $x(y)$ of this ball. Moreover, in the ball $B(x_0, \rho'/2)$, for sufficiently small $|y|$, the inequality holds

$$\left[\sum_{i,j=1}^n a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i} \right] v_\varepsilon(x + y) > f(x) \quad \text{for } |x - x_0| < \rho'/2.$$

Thus, by Lemma 1, $\nabla v_\varepsilon[x(y) + y] = 0$, $u[x(y)] = 0$. Using the second of inequalities (5), we find that $x(y) + y = x_0$, or $x(y) = x_0 - y$. Consequently, $u \equiv 0$ in a neighborhood of the point x_0 , which contradicts Lemma 1.

- b) Suppose now that there exists a sequence of points x^k satisfying the conditions $x^k \rightarrow x_0$ as $k \rightarrow \infty$, $v(x^k) > v(x_0)$.

Choose $\varepsilon > 0$ so small that the point x_0 of maximum of the difference $v - u$ is contained in the ball $B(x', \rho - \varepsilon)$, and on the boundary of this ball (see condition 2)) the inequality $|v(x) - u(x)| < |v(x_0) - u(x_0)|$ holds. Introduce the sequence of functions $v_k(x) = v(x + \xi_k)$, $\xi_k = x^k - x_0$. For large k , the function $v_k(x)$ in the ball $B(x', \rho - \varepsilon)$ satisfies the inequality $Lv_k(x) > f(x)$, and the maximum of the difference $v_k - u$ in this ball is attained at an interior point x_0^k . By Lemma 1,

$$\nabla v_k(x_0^k) = \nabla v(x_0^k + \xi_k) = 0, \quad v_k(x_0^k) - u(x_0^k) = v(x_0^k + \xi_k). \quad (6)$$

We now note that at the point x_0 the difference $v_k - u$ takes the value $v(x_0 + \xi_k) = v(x^k)$, and therefore

$$v(x_0^k + \xi_k) \geq v(x^k) > v(x_0). \quad (7)$$

Moreover, by continuity (with respect to ξ) of the maximum of the difference $v(x + \xi) - u(x)$,

$$v(x_0^k + \xi_k) \rightarrow v(x_0) \quad \text{as } k \rightarrow \infty. \quad (8)$$

From (6), (7), and (8) there follows the existence of a sequence of points $\bar{x}_k = x_0^k + \xi_k$, $k \geq k_0$, with the following properties: $\bar{x}_k \in B(x', \rho - \varepsilon/2)$; $\nabla v(\bar{x}_k) = 0$; $v(\bar{x}_k) > v(x_0)$; $v(\bar{x}_k) \rightarrow v(x_0)$ as $k \rightarrow \infty$. The existence of such a sequence of points for the analytic function $v(x)$ in the ball $B(x', \rho)$ contradicts Lemma 2.

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Note: Figure translations are in progress. See original paper for figures.

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