

INVESTIGATION OF THE DEFECT OF AN ABSTRACT CAUCHY PROBLEM

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Abstract

Full Text

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MATHEMATICS

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INVESTIGATION OF THE DEFECT OF AN ABSTRACT CAUCHY PROBLEM

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Consider the abstract Cauchy problem

$$dx(t)/dt = Ax(t) \quad (0 < t < \infty), \quad x(0) = x_0, \quad (1)$$

where A is a closed linear operator in a Banach space $\mathfrak{B}$. The set of those vectors x_0 for which problem (1) has a solution* (strongly continuously differentiable for $t > 0$ and strongly continuous for $t = 0$) is called the initial manifold and is denoted by I_A . In [1] some criteria are given for the density of the manifold I_A in the whole space under the condition that the domain D_A of the operator A is dense. If D_A is not dense, then the manifold I_A is all the more not dense, as is shown by the following simple

Lemma 1. *The inclusion always holds** *

$$\bar{I}_A \subseteq \bar{D}_A. \quad (2)$$

Indeed, if $x_0 \in I_A$, then $x_0 = \lim_{t \rightarrow 0} x(t)$, where $x(t)$ is the corresponding solution. But $x(t) \in D_A$ ($t > 0$).

Relation (2) gives reason to introduce the following general

Definition. The dimension of the quotient space $\bar{D}_A/\bar{I}_A$ is called the defect of the Cauchy problem (1) and is denoted by d_A .

Now the subject of the present article is clear from its title.

Recall that, for any subspace $\mathcal{L}$, the dimension of the quotient space $\mathfrak{B}/\mathcal{L}$ is called the defect of the subspace $\mathcal{L}$ and is denoted by $\text{def } \mathcal{L}$.

Lemma 2. *If the resolvent set of the operator A is nonempty, then the inequality holds*

$$\dim(\bar{D}_{A^k}/\bar{D}_{A^{k+1}}) \leq \text{def } \bar{D}_A \quad (k = 1, 2, \dots). \quad (3)$$

Proof. If $\{f_i\}_1^p$ is a linearly independent system of linear functionals on $\overline{D}_{A^k}$, orthogonal to $\overline{D}_{A^{k+1}}$, then $\{(R_\mu^k)^* f_i\}_1^p$ is a linearly independent system of linear functionals on all of $\mathfrak{B}$, orthogonal to $\overline{D}_A$. Here R_μ denotes, as usual, the resolvent of the operator A , i.e. $R_\mu = (A - \mu E)^{-1}$; μ is some point of the resolvent set.

We can now establish the following central result:

Theorem 1. *Let some half-plane $\operatorname{Re} \lambda \geq a$ belong to the resolvent set of the operator A , and in this half-plane*

$$\|R_\lambda\| = O(1 + |\lambda|^m) \quad (4)$$

for some $m \geq 0$. Then**

$$d_A \leq [m + 1] \operatorname{def} \overline{D}_A. \quad (5)$$

* Not necessarily unique.

** The bar denotes closure.

*** $[\]$ denotes the integer part.

This theorem is, in essence, a generalization of Theorem 2 ⁽¹⁾ and is proved in a similar way.*

Proof. Let $\operatorname{Re} \mu > \alpha$. Take an arbitrary vector x and put

$$u(t) = -\frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{\lambda t} R_\lambda x}{(\lambda - \mu)^{[m+2]}} d\lambda \quad (t \geq 0).$$

The vector-function $u(t)$ is strongly continuous. Put

$$u_h(t) = \frac{1}{h} \int_t^{t+h} u(\tau) d\tau = \frac{1}{2\pi i} \int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{\lambda h} - 1}{\lambda} \frac{e^{\lambda t} R_\lambda x}{(\lambda - \mu)^{[m+2]}} d\lambda \quad (h > 0).$$

The vector-function $u_h(t)$, for every $h > 0$, is a solution of problem (1) with $x_0 = u_h(0)$. Therefore $u_h(0) \in I_A$. Consequently, $u(0) \in \overline{I}_A$. But $u(0) = R_\mu^{[m+2]} x$. In view of the arbitrariness of the vector x ,

$$\overline{D}_{A^{[m+2]}} \subseteq \overline{I}_A.$$

Hence, by Lemma 2,

$$d_A \leq \dim(\overline{D}_A / \overline{D}_{A^{[m+2]}}) = \sum_{k=1}^{[m+1]} \dim(\overline{D}_{A^k} / \overline{D}_{A^{k+1}}) \leq [m+1] \operatorname{def} \overline{D}_A,$$

as was required.

Theorem 1 is sharp first of all in the sense that the following is true.

Theorem 2. *For any integers $m \geq 0$, $p > 0$, there exists an operator A satisfying the conditions of Theorem 1 such that*

$$\operatorname{def} \overline{D}_A = p, \quad d_A = (m+1)p. \quad (6)$$

Moreover, the following is true.

Theorem 3. *Let $\rho(r) > 1$ ($r \geq 0$) be an arbitrary function growing faster than any power in the sense that $r^{-m}\rho(r) \rightarrow \infty$ ($r \rightarrow \infty$), whatever $m > 0$. Then there exists an operator A , having no spectrum in the half-plane $\operatorname{Re} \lambda \geq 0$, such that*

$$\|R_\lambda\| \leq \rho(|\lambda|) \quad (\operatorname{Re} \lambda \geq 0) \quad (7)$$

and at the same time

$$\operatorname{def} \overline{D}_A = 1, \quad d_A = \infty.$$

The proof of Theorem 2 is reduced to the case $p = 1$ with the aid of the following simple proposition.

Lemma 3. *Let closed linear operators A_1, A_2 act in Banach spaces $\mathfrak{B}_1, \mathfrak{B}_2$, respectively. Form the direct sums $\mathfrak{B} = \mathfrak{B}_1 + \mathfrak{B}_2$, $A = A_1 + A_2$. Then*

$$\operatorname{def} \overline{D}_A = \operatorname{def} \overline{D}_{A_1} + \operatorname{def} \overline{D}_{A_2}, \quad d_A = d_{A_1} + d_{A_2}. \quad (8)$$

Proof of Theorem 2 for the case $p = 1$. Let $\mathfrak{B} = C^{m+1}[0, 1]$ —the space of functions $\varphi(s)$ ($0 \leq s \leq 1$) having a continuous $(m+1)$ -st derivative;

$$\|\varphi\| = \max_{0 \leq n \leq m+1} \left(\max_{0 \leq s \leq 1} |\varphi^{(n)}(s)| \right).$$

Let A be the differentiation operator $-d/ds$ on functions having a continuous $(m+2)$ -nd derivative and satisfying the boundary condition $\varphi(0) = 0$. Obviously, $\operatorname{def} \overline{D}_A = 1$.

The operator A has no spectrum: for any $\psi \in \mathfrak{B}$ and any λ ,

$$R_\lambda \psi(s) = -e^{-\lambda s} \int_0^s \psi(\tau) e^{\lambda \tau} d\tau \quad (0 \leq s \leq 1).$$

* Theorem 2 (1) corresponds to the case $\text{def } \overline{D}_A = 0$. Then $d_A = 0$.

Hence, by integration by parts, one obtains the estimate

$$\max_{0 \leq s \leq 1} |R_\lambda \psi(s)| \leq \frac{1}{|\lambda|} \left(2 \max_{0 \leq s \leq 1} |\psi(s)| + \max_{0 \leq s \leq 1} |\psi'(s)| \right) \quad (\text{Re } \lambda > 0). \quad (9)$$

But, since

$$(R_\lambda \psi(s))^n = -\lambda (R_\lambda \psi(s))^{(n-1)} - \psi^{(n-1)}(s)$$

$$(n = 1, 2, \dots, m+1; 0 \leq s \leq 1),$$

it follows, in view of (9), that

$$\max_{0 \leq s \leq 1} |(R_\lambda \psi(s))^{(n)}| = O(\|\psi\|(1 + |\lambda|^{n-1}))$$

$$(n = 0, 1, \dots, m+1; \text{Re } \lambda > 0),$$

i.e. condition (4) is satisfied in the half-plane $\text{Re } \lambda > 0$.

At the same time, the solution of problem (1) with initial function $x_0 = \xi(s)$ has the form

$$x(t) = \xi(s-t) \max(0, \text{sign}(s-t)).$$

In order that $\xi(s) \in L_A$, it is necessary and sufficient that

$$\xi^{(n)}(0) = 0 \quad (n = 0, 1, \dots, m+2).$$

Therefore

$$\Gamma_A = \{\xi \mid \xi^{(n)}(0) = 0 \ (n = 0, 1, \dots, m+1)\},$$

whence it is clear that $d_A = m+1$.

We now prove Theorem 3. Let $\{M_n\}_0^\infty$ be a sequence of positive numbers ($M_0 = 1, M_1 \geq 1$) satisfying the conditions

$$\sum_{k=0}^{\infty} \frac{r^k}{M_k} \leq \rho(r) \quad (r \geq 0); \quad M_{p+q} \geq M_p M_q \quad (p, q = 0, 1, 2, \dots). \quad (10)$$

It is easy to construct it inductively. Consider the Banach space $C\{M_n\}$ of infinitely differentiable functions $\varphi(s)$ ($0 \leq s \leq 1$) with norm

$$\|\varphi\| = \sup_n \frac{1}{M_n} \max_{0 \leq s \leq 1} |\varphi^{(n)}(s)|.$$

In this space, take the operator $\tilde{A} = -d/ds$ on those functions $\varphi(s)$ for which $\varphi'(s) \in C\{M_n\}$, $\varphi(0) = 0$.

Let $\psi(s) \in C\{M_n\}$. Then the equation

$$-\varphi'(s) - \lambda\varphi(s) = \psi(s) \quad (0 \leq s \leq 1)$$

under the condition $\varphi(0) = 0$ has the unique solution

$$\varphi(s) = -e^{-\lambda s} \int_0^s \psi(\tau) e^{\lambda \tau} d\tau. \quad (11)$$

This solution belongs to $C\{M_n\}$ in the half-plane $\operatorname{Re} \lambda \geq 0$, since, recursively,

$$\max_{0 \leq s \leq 1} |\varphi^{(n)}(s)| \leq |\lambda|^n \max_{0 \leq s \leq 1} |\psi(s)| + \sum_{k=1}^n |\lambda|^{n-k} \max_{0 \leq s \leq 1} |\psi^{(k-1)}(s)| \quad (n = 0, 1, 2, \dots),$$

whence, by (10),

$$\|\varphi\| \leq \|\psi\| \sum_{k=0}^n \frac{|\lambda|^k}{M_k} \leq \rho(|\lambda|) \|\psi\|.$$

We see that the operator $\tilde{A}$ has no spectrum in the half-plane $\operatorname{Re} \lambda \geq 0$, and its resolvent satisfies inequality (7). When is $\operatorname{def} \bar{D}_{\tilde{A}} = 1$? This question appears difficult (see, for example, (2), pp. 240-243). We shall avoid it by considering, instead of the operator $\tilde{A}$, a certain restriction $\tilde{A}$. Let Π be the linear manifold of all polynomials. Obviously, $\Pi \subset C\{M_n\}$. Put

$$\Pi_0 = \Pi \cap \{\varphi \mid \varphi(0) = 0\}.$$

The operator $\tilde{A}$ everywhere on Π_0

is defined and maps Π_0 into Π . Restrict it to Π_0 and close it. This gives a certain operator $A \subset \tilde{A}$, acting in the Banach space $\mathfrak{B} = \bar{\Pi} \subset C\{M_n\}$. Obviously, $\bar{D}_A = \bar{\Pi}_0$ and, therefore, $\operatorname{def} \bar{D}_A = 1$. Further, if $\xi(s) \in I_A$, then $\xi^{(n)}(0) = 0$ ($n = 0, 1, 2, \dots$) and, consequently, $d_A = \infty$. It remains to verify that, when the operator $\tilde{A}$ is restricted to the operator A , no residual spectrum appears. For this it is enough to prove that if in (11) the function ψ is a polynomial, then $\varphi \in \Pi$. This assertion, in turn, reduces to the inclusions

$s^k e^{-\lambda s} \in \overline{\Pi}$ ($k = 0, 1, 2, \dots; \operatorname{Re} \lambda \geq 0$). But the point is that the expansion of $e^{-\lambda s}$ in a Taylor series, as is not difficult to verify, converges in the space $C\{M_n\}$, and the expansions of the remaining functions $s^k e^{-\lambda s}$ are obtained by repeated differentiation with respect to λ . Thus Theorem 3 is proved.

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Note: Figure translations are in progress. See original paper for figures.

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