

# ON SOME IDENTITIES OF THE GENERAL THEORY OF ORTHOGONAL POLYNOMIALS AND THEIR APPLICATIONS

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1966

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## Abstract

## Full Text

UDC 517.512.6+517.512.7

## MATHEMATICS

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# ON SOME IDENTITIES OF THE GENERAL THEORY OF ORTHOGONAL POLYNOMIALS AND THEIR APPLICATIONS

(Presented by Academician S. N. Bernstein, 4 I 1966)

**1°.** Let  $\{\omega_k(x)\}_{k=0}^{\infty}$  be a system of polynomials orthonormal on the interval  $[-1, 1]$  with respect to the weight  $g(x)$ , and let  $a_n$  be the leading coefficient of  $\omega_n(x)$ . It is known that the recurrence formula holds

$$\omega_k(x) = (\alpha_k x + \beta_k) \omega_{k-1}(x) - \gamma_k \omega_{k-2}(x), \quad k = 2, 3, \dots, \quad (1)$$

where  $\alpha_k, \beta_k, \gamma_k$  are constants, with  $\alpha_k = a_k/a_{k-1}$ ,  $\gamma_k = \alpha_k/\alpha_{k-1}$ .

Of importance in the general theory of orthogonal polynomials is the Christoffel-Darboux formula

$$\sum_{k=0}^n \omega_k(x) \omega_k(t) = \frac{a_n}{a_{n+1}} \frac{\omega_{n+1}(t) \omega_n(x) - \omega_n(t) \omega_{n+1}(x)}{t - x}. \quad (2)$$

In the present note we study the sum  $\sum_{k=0}^n \omega'_k(x) \omega'_k(t)$ , where the prime, as usual, denotes the first derivative. We shall show that this sum is also of interest.

**2°.** **Theorem 1.** The identity holds

$$\begin{aligned} \sum_{k=0}^n \omega'_k(x) \omega'_k(t) &= \frac{a_n}{a_{n+1}} \left\{ \frac{\omega'_{n+1}(t) \omega'_n(x) - \omega'_n(t) \omega'_{n+1}(x)}{t - x} + \right. \\ &+ (t - x)^{-3} [(\omega'_{n+1}(t) \omega_n(x) - \omega'_n(t) \omega_{n+1}(x) + \omega'_{n+1}(x) \omega_n(t) \\ &\left. - \omega'_n(x) \omega_{n+1}(t))(t - x) + 2(\omega_{n+1}(x) \omega_n(t) - \omega_{n+1}(t) \omega_n(x))] \right\}. \quad (3) \end{aligned}$$

We shall outline the proof. Differentiate equality (1) and then multiply it by  $\omega_{k-1}(t)$ . In the resulting equality replace  $x$  by  $t$  and  $t$  by  $x$ , and subtract the

second equality from the first. The subsequent reasoning is the same as in the derivation of the Christoffel-Darboux formula. The only difference is that on the right-hand side of the equality there appear sums of the form  $\sum_{k=0}^n \omega'_k(x) \omega_k(t)$  and  $\sum_{k=0}^n \omega_k(x) \omega'_k(t)$ , which are found by differentiating formula (2), respectively with respect to  $x$  and to  $t$ .

**Theorem 2.** The identity holds

$$\sum_{k=0}^n (\omega'_k(x))^2 = \frac{a_n}{a_{n+1}} \left[ \frac{1}{2} (\omega''_{n+1}(x) \omega'_n(x) - \omega''_n(x) \omega'_{n+1}(x)) + \right. \\ \left. + \frac{1}{6} (\omega'''_{n+1}(x) \omega_n(x) - \omega'''_n(x) \omega_{n+1}(x)) \right].$$

**Proof.** This theorem is easily derived from Theorem 1. It suffices in equality (3) to put  $t = x$  and then compute its right-hand side by L' Hôpital' s rule.

**Corollary 1.** For any orthonormal system of polynomials  $\{\omega_k(x)\}_{k=0}^\infty$  the inequality holds

$$\omega'''_{n+1}(x) \omega_n(x) + 3\omega''_{n+1}(x) \omega'_n(x) \geq \omega'''_n(x) \omega_{n+1}(x) + 3\omega''_n(x) \omega'_{n+1}(x), \\ n = 0, 1, 2, \dots, \quad -\infty < x < \infty.$$

**Theorem 3.** For any orthonormal system of polynomials  $\{\omega_k(x)\}_{k=0}^\infty$  with weight  $g(x)$ , the equality holds

$$\sum_{k=0}^n \int_{-1}^1 (\omega'_k(x))^2 g(x) dx = \frac{a_n}{2a_{n+1}} \int_{-1}^1 [\omega''_{n+1}(x) \omega'_n(x) - \omega''_n(x) \omega'_{n+1}(x)] g(x) dx.$$

**Proof.** Since  $\omega'''_{n+1}(x)$ ,  $\omega'''_n(x)$  are polynomials of degrees respectively  $[n-2]$  and  $(n-3)$ , by orthogonality,

$$\int_{-1}^1 \omega'''_{n+1}(x) \omega_n(x) g(x) dx = \int_{-1}^1 \omega'''_n(x) \omega_{n+1}(x) g(x) dx = 0.$$

Therefore Theorem 3 follows from Theorem 2.

**Corollary 2.** For any orthonormal system of polynomials  $\{\omega_k(x)\}_{k=0}^\infty$  with weight  $g(x)$ , the inequality holds

$$\int_{-1}^1 \omega''_{n+1}(x) \omega'_n(x) g(x) dx \geq \int_{-1}^1 \omega''_n(x) \omega'_{n+1}(x) g(x) dx, \quad n = 0, 1, 2, \dots$$

**Theorem 4.** For the orthonormal system of Legendre polynomials  $\{p_k(x)\}_{k=0}^\infty$ , the equality holds

$$\sum_{k=0}^n \int_{-1}^1 (p'_k(x))^2 dx = \frac{n(n+1)^2(n+2)}{4}, \quad n = 0, 1, 2, \dots$$

We indicate the proof. From the orthogonality of the Legendre polynomials and the formula of integration by parts it follows that

$$\int_{-1}^1 [p_{n+1}''(x)p_n'(x) - p_n''(x)p_{n+1}'(x)] dx = [p_n(x)p_{n+1}''(x) - p_{n+1}(x)p_n''(x)]_{-1}^1.$$

It is not difficult to calculate that the right-hand side of this equality is equal to

$$\frac{n(n+1)(n+2)}{2} \sqrt{(2n+1)(2n+3)}.$$

Therefore Theorem 4 follows from Theorem 3.

3°. With the help of Theorem 4 one can obtain an analogue of A. A. Markov's inequality for the space  $L_2$  of all functions square-summable on the interval  $[-1, 1]$  with norm

$$\|f\|_{L_2} = \left( \int_{-1}^1 f^2 dx \right)^{1/2}.$$

**Theorem 5.** For any polynomial  $R_n$  of degree  $n$ , the inequality holds

$$\|R_n'\| \leq \frac{n+1}{2} \sqrt{n(n+2)} \|R_n\|_{L_2}, \quad n = 0, 1, 2, \dots \quad (4)$$

**Proof.** Consider the operator

$$U_n(f, x) = \int_{-1}^1 f(t) \sum_{j=0}^n p_j'(x) p_j(t) dt, \quad f \in L_2.$$

It is clear that

$$U_n(R_n, x) = R_n'(x).$$

Therefore, from the Cauchy–Bunyakovsky inequality and the orthogonality of the Legendre polynomials it follows that

$$\|R_n'\|_{L_2} \leq \left( \sum_{k=0}^n \int_{-1}^1 (p_k'(x))^2 dx \right)^{1/2} \|R_n\|. \quad (5)$$

By Theorem 4 and this inequality we have (4).

**Remark 1.** For  $n = 1$  and  $R_1(x) = p_1(x)$ , equality holds in (4).

**Remark 2.** If the norm is defined by the formula

$$\|f\| = \left( \int_{-1}^1 f^2 g dx \right)^{1/2},$$

then the preceding arguments lead to the inequality

$$\|R_n^{(k)}\|_{L_2} \leq \left( \sum_{j=0}^n \int_{-1}^1 (\omega_j^{(k)}(x))^2 g(x) dx \right)^{1/2} \|R_n\|,$$

which is a generalization of inequality (5) to the general case of orthogonal polynomials.

4°. Let  $\bar{\Omega}_n^{(k)}(L_2)$  be the set of all linear operations  $V(f)$  from  $L_2$  into  $L_2$  having the property that  $V(f, x) = f^{(k)}(x)$ , if  $f$  is a polynomial of degree  $\leq n$ , and let  $\bar{\Omega}_{n,n}^{(k)}(L_2)$  be the set of all linear operations  $V(f)$  from  $L_2$  into  $L_2$  satisfying the conditions: 1) for any  $f \in L_2$ ,  $V(f)$  is a polynomial of degree  $\leq n$ ; 2) if  $f$  is a polynomial of degree  $\leq n$ , then  $V(f, x) = f^{(k)}(x)$ . Obviously,  $\bar{\Omega}_{n,n}^{(k)}(L_2) \subset \bar{\Omega}_n^{(k)}(L_2)$ . It can be shown that  $\bar{\Omega}_n^{(k)}(L_2) \neq \bar{\Omega}_{n,n}^{(k)}(L_2)$ . Put

$$\bar{\rho}_{n,n}^{(k)} = \rho_{n,n}^{(k)}(L_2) = \inf_{V \in \bar{\Omega}_{n,n}^{(k)}(L_2)} \|V\|, \quad \bar{\rho}_n^{(k)} = \rho_n^{(k)}(L_2) = \inf_{V \in \bar{\Omega}_n^{(k)}(L_2)} \|V\|.$$

In the space  $C$  of all continuous  $2\pi$ -periodic functions with norm  $\|f\| = \max_{0 \leq x < 2\pi} |f(x)|$ , the equality (1)\* holds

$$\lim_{n \rightarrow \infty} \left( \frac{\rho_{n,n}^{(k)}}{\rho_n^{(k)}} : \frac{4}{\pi^2} \ln n \right) = 1.$$

In the space  $C$  of all continuous functions on the interval  $[-1, 1]$ , with norm  $\|f\| = \max_{-1 \leq x \leq 1} |f(x)|$ ,  $\bar{\rho}_n^{(k)} = \bar{\rho}_{n,n}^{(k)} = T_n^{(k)}(1)$ ,  $k = 1, 2, \dots, n$ ,  $T_n(x) = \cos n \arccos x$  (2). A. N. Kolmogorov drew my attention to the fact that in the space  $L_2$  of all  $2\pi$ -periodic square-integrable functions with norm

$$\|f\| = \left( \int_0^{2\pi} f^2 dx \right)^{1/2}$$

the equalities  $\rho_{n,n}^{(k)} = \rho_n^{(k)} = n^k$ ,  $k = 0, 1, 2, \dots$ , hold. We shall now prove a theorem concerning the space  $L_2$ .

**Theorem 6.** For any  $0 \leq k \leq n$ ,

$$\|p_n^{(k)}\| \leq \bar{\rho}_n^{(k)}(L_2) \leq \bar{\rho}_{n,n}^{(k)}(L_2) \leq \|\Pi_n\|, \quad (6)$$

$$\Pi_n(x) = \left[ \sum_{j=0}^n (p_j^{(k)}(x))^2 \right]^{1/2},$$

\* For the periodic case the bar over  $\rho$  is omitted.

where  $\{p_j(x)\}_{j=0}^{\infty}$  are the orthonormal Legendre polynomials. In particular,

$$\bar{\rho}_n^{(n)}(L_2) = \rho_{n,n}^{(n)}(L_2) = (2n-1)!!\sqrt{2n+1}. \quad (7)$$

**Proof.** It is obvious that the operation

$$\bar{V}(f, x) = \int_{-1}^1 f(t) \sum_{j=0}^n p_j^{(k)}(x) p_j(t) dt$$

is of the class  $\bar{\Omega}_{n,n}^{(k)}(L_2)$ , and  $\|\bar{V}\| \leq \|\Pi_n\|$ . Therefore

$$\bar{\rho}_n^{(n)}(L_2) \leq \bar{\rho}_{n,n}^{(n)}(L_2) \leq \|\Pi_n\|. \quad (8)$$

On the other hand, for any  $V \in \bar{\Omega}_n^{(k)}(L_2)$ ,  $V(p_n) = p_n^{(k)}$ , and since  $\|p_n\| = 1$ , it follows that  $\bar{\rho}_n^{(k)}(L_2) \geq \|p_n^{(k)}\|$ . Hence, together with (8), (6) follows.

If  $k = n$ , then, by virtue of (6), we have

$$\bar{\rho}_n^{(n)}(L_2) = \bar{\rho}_{n,n}^{(n)}(L_2) = p_n^{(n)}(x)\sqrt{2},$$

and this is equivalent to the equalities (7).

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Received  
9 XII 1965

## REFERENCES

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- <sup>2</sup> D. L. Berman, DAN, **140**, No. 3 (1961).

*Note: Figure translations are in progress. See original paper for figures.*

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