

# ON A CRITERION FOR THE CONVERGENCE OF THE GRADIENTS OF A SEQUENCE OF HARMONIC FUNCTIONS

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## Abstract

## Full Text

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*MATHEMATICS*

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# ON A CRITERION FOR THE CONVERGENCE OF THE GRADIENTS OF A SEQUENCE OF HARMONIC FUNCTIONS

In the classical theory of functions, an important role is played by the following theorem on the convergence of a sequence  $\{f_n(z)\}$ ,  $n = 1, 2, \dots$ , of functions holomorphic in a domain  $D$  of the plane of the complex variable  $z = x + iy$ : if the sequence  $\{u_n(x, y) = \operatorname{Re} f_n(z)\}$  converges to zero uniformly in the domain  $D$ , and the sequence  $\{v_n(x, y) = \operatorname{Im} f_n(z)\}$  converges to zero at a fixed point  $z_0 \in D$ , then  $\{f_n(z)\}$  converges to zero uniformly in every bounded closed domain  $D^*$  belonging to the domain  $D$ .

The present note is devoted to establishing a multidimensional analogue of this theorem.

For simplicity of notation, below we shall restrict ourselves to consideration of the three-dimensional case.

If a sequence  $\{u_n(x, y, z)\}$ ,  $n = 1, 2, \dots$ , of harmonic functions regular in a domain  $D$  of the space of the variables  $x, y, z$  has the properties: a) the sequence  $\{\partial u_n / \partial x\}$  converges to zero in the domain  $D$  uniformly with respect to the variables  $x, y, z$ ; b) the sequence  $\{\partial u_n / \partial y\}$  converges to zero in the domain  $D$  uniformly with respect to the variables  $y, z$ , and c) the sequence  $\{\partial u_n / \partial z\}$  converges to zero at a fixed point of the domain  $D$ , for example, at the point  $(0, 0, 0)$ , then the sequence  $\{\operatorname{grad} u_n(x, y, z)\}$  converges to zero uniformly with respect to  $x, y, z$  in every bounded closed domain  $D^*$  lying in the domain  $D$ .

For the purpose of proving our assertion, let us note that it is always possible to indicate a positive number  $r$  such that the closed ball  $C(r; x_0, y_0, z_0)$  of radius  $r$  with center at any point  $(x_0, y_0, z_0) \in D^*$  will lie in the domain  $D$ .

Inside the ball  $C(r; x_0, y_0, z_0)$ , for each harmonic function of the sequence  $\{u_n(x, y, z)\}$  the integral representation  $(1, 2)$  holds

$$u_n(x, y, z) = \frac{1}{4\pi r^2} \iint_S \left[ \frac{(r^2 - \xi^2 - \eta^2 - \zeta^2)(x - \xi)}{\Delta R^{1/2}} + \frac{x + \xi}{R^{1/2}} - \operatorname{Arsh} \frac{x - \xi}{\Delta^{1/2}} \right] \times$$

$$\times \frac{\partial u_n(\xi, \eta, \zeta)}{\partial \xi} dS + \gamma_n(y, z; x_0, y_0, z_0), \quad (1)$$

where  $S$  is the sphere

$$(\xi - x_0)^2 + (\eta - y_0)^2 + (\zeta - z_0)^2 = r^2,$$

$$\Delta = (y - \eta)^2 + (z - \zeta)^2, \quad R = (x - \xi)^2 + \Delta,$$

and  $\gamma_n(y, z; x_0, y_0, z_0)$  is a completely determined regular harmonic function of the variables  $y, z$  in the cylinder

$$(y - y_0)^2 + (z - z_0)^2 < r^2.$$

Denote by  $\delta$  and  $\delta_1$  positive numbers satisfying the conditions  $\delta < \delta_1 < r$ .

Computing the partial derivatives  $\partial u_n / \partial y$  and  $\partial u_n / \partial z$  from formula (1) at the points  $(0, y, z)$  and  $(0, 0, 0)$ , respectively, by virtue of conditions a), b), and c), we conclude that the sequence  $\{\partial \gamma_n(y, z; 0, 0, 0) / \partial y\}$  converges to zero uniformly for  $y^2 + z^2 \leq \delta_1^2$ , and the sequence  $\{\partial \gamma_n(y, z; 0, 0, 0) / \partial z\}$  converges to zero for  $y = z = 0$ . Hence, by virtue of the theorem formulated at the beginning of the present note, there follows uniform convergence to zero

of the sequence  $\{\text{grad } \gamma_n(y, z; 0, 0, 0)\}$  for  $y^2 + z^2 \leq \delta^2$ . Taking this circumstance into account, on the basis of conditions a), b), and c), again from formula (1) we conclude that the sequence  $\{\text{grad } u_n(x, y, z)\}$  converges uniformly to zero in the ball  $C(\delta; 0, 0, 0)$ .

Moving the center of the sphere  $C(r; 0, 0, 0)$  to the point  $(x_0, y_0, z_0)$  along a continuous path  $L$  lying in the domain  $D^*$ , and taking into account that condition c) may be regarded as fulfilled at the point  $(x_0, y_0, z_0)$ , by repeating the argument just given we become convinced of the uniform convergence to zero of the sequence  $\{\text{grad } u_n(x, y, z)\}$  in the sphere  $C(\delta; x_0, y_0, z_0)$ . Hence, by virtue of the Heine-Borel lemma, the validity of our assertion follows immediately.

The assertion proved is, in an obvious way, rephrased for a sequence of vectors  $\{P_n(u_n, v_n, w_n)\}$  that are regular solutions of the system  $\text{div } P_n = 0$ ,  $\text{rot } P_n = 0$ .

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## REFERENCES

1. A. V. Bitsadze, DAN, **159**, No. 5 (1965).

2. A. V. Bitsadze, *Boundary Value Problems for Elliptic Equations of the Second Order*, "Nauka," 1966.

*Note: Figure translations are in progress. See original paper for figures.*

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