

ON REPRESENTATIONS OF COMMUTATIVE SYMMETRIC BANACH ALGEBRAS AND COMMUTATIVE TOPOLOGICAL GROUPS IN THE SPACE $\bigwedge(\bigwedge \Pi_k)$

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Abstract

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MATHEMATICS

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ON REPRESENTATIONS OF COMMUTATIVE SYMMETRIC BANACH ALGEBRAS AND COMMUTATIVE TOPOLOGICAL GROUPS IN THE SPACE Π_k

(Presented by Academician I. M. Vinogradov on 27 XII 1965)

1. Let R be a commutative separable Banach algebra with identity e ; we shall call the algebra R **symmetric** if an involution $x \rightarrow x^*$ is defined in R satisfying the conditions: 1) $x^{**} = x$; 2) $(\lambda x)^* = \bar{\lambda}x^*$, λ a number; 3) $(x + y)^* = x^* + y^*$; 4) $(xy)^* = x^*y^*$; 5) $|x^*| = |x|$. By a **representation** of the algebra R in the Pontryagin space Π_k we shall mean such a homomorphism $x \rightarrow A_x$ of the algebra R into the algebra of bounded linear operators in Π_k that: α) $|A_x| \leq C|x|$ for all $x \in R$ and some constant C , where $|A_x|$ is the norm of the operator A_x ; β) $A_{x^*} = (A_x)^*$, where $(A_x)^*$ is the operator adjoint to A_x in the sense of the indefinite scalar product (x, y) in Π_k .

The aim of the present article is to describe all representations in Π_k of the algebra R ; as a consequence one obtains a description of all unitary (in the indefinite sense) representations in Π_k of a commutative locally bicomact group G with a countable base of neighborhoods. We note that in the author's papers (^{1,2}) a more general case of noncommutative algebras and groups was considered; however, the commutative case admits further detail.

In what follows the term "algebra" will denote a commutative symmetric separable Banach algebra with identity e ; moreover, everywhere below the space Π_k is assumed to be separable.

2. We first describe representations $x \rightarrow A_x$ of the algebra R in an ordinary separable Hilbert space $\mathfrak{H}$, where now in β) $(A_x)^*$ denotes the usual adjoint operator. Let T be the bicomact space of symmetric maximal ideals t of the algebra R , and let $x(t)$ be the value of the element $x \in R$ on the ideal t ; applying I, item 4 § 17 and Theorems 1 and 2 of item 3 (see the proof of Theorem 2) in (³), we arrive at the following result:

Theorem 1. *Every symmetric representation $x \rightarrow A_x$ of the algebra R in a*

separable Hilbert space $\mathfrak{H}$ is equivalent to a representation in the direct integral

$$\mathfrak{H} = \int_T \mathfrak{H}(t) d\sigma$$

with respect to some Borel measure σ on T of a σ -measurable family of separable Hilbert spaces $\mathfrak{H}(t)$ and is given by the formula

$$A_x \{h(t)\} = \{x(t)h(t)\}$$

for $\{h(t)\} \in \mathfrak{H}$, $h(t) \in \mathfrak{H}(t)$.

3. Let now $x \rightarrow A_x$ be a representation of the algebra R in the space Π_k . Put $\mathfrak{R} = \{A_x : x \in R\}$; then $\mathfrak{R}$ is a commutative symmetric algebra of operators in Π_k (see ⁽⁴⁾ or ⁽⁵⁾). Its proper functionals (p.f.) $\lambda_j(A_x)$ are called the p.f. of the representation $x \rightarrow A_x$ and are denoted by $\lambda_j(x)$, while the hyperbolic subspace H of the algebra $\mathfrak{R}$ is called the **hyperbolic subspace** of the representation $x \rightarrow A_x$. If $H \neq (0)$, then from the results of item 2 of ⁽⁴⁾ (see also item 3 of ⁽⁵⁾) it follows:

Theorem 2. *The restriction of the representation of the algebra R to the hyperbolic subspace is the orthogonal sum of representations, each of which*

which are a direct sum of two mutually conjugate representations on two skew-related null subspaces.

Since $\Pi_k = H \oplus H^\perp = H \dot{+} H^\perp$, and the restriction of the representation $x \rightarrow A_x$ has no non-real characters on $H^\perp$, it suffices to restrict ourselves to considering representations $x \rightarrow A_x$ in Π_k with only real characters, which, evidently, may be regarded as points of the space T .

Let, in this case, $\mathfrak{P}$ be some k -dimensional nonnegative subspace invariant with respect to all A_x , $x \in R$. Put $\mathfrak{M} = \mathfrak{P}^\perp$, $\mathfrak{R} = \mathfrak{P} \cap \mathfrak{M}$; then $\mathfrak{M}, \mathfrak{R}$ are also invariant with respect to all A_x , $x \in R$, $\mathfrak{M}$ is a nonpositive, and $\mathfrak{R}$ a null subspace in Π_k ; choose $\mathfrak{P}$ so that $\mathfrak{R}$ has maximal dimension. Let $\mathfrak{R}'$ be a null subspace in Π_k skew-related to $\mathfrak{R}$; put $\mathfrak{H} = \mathfrak{M} \cap \mathfrak{R}'^\perp$, $\Pi = \mathfrak{P} \cap \mathfrak{R}'^\perp$. Then

$$\mathfrak{M} = \mathfrak{R} \oplus \mathfrak{H}, \quad \mathfrak{P} = \mathfrak{R} \oplus \Pi, \quad \Pi_k = (\mathfrak{R} \dot{+} \mathfrak{R}') \oplus \mathfrak{H} \oplus \Pi, \quad (1)$$

where $\mathfrak{R}, \mathfrak{R}', \mathfrak{H}, \Pi$ are linearly independent, Π is positive, $\mathfrak{H}$ is negative, i.e. a Hilbert space with scalar product $[x, y] = -(x, y)$. Let $\mathfrak{S}_\lambda$ be the root lineal of the algebra $\mathfrak{R}$ corresponding to the character λ , and let $\lambda_1, \dots, \lambda_q$ be those among all the characters $\lambda_1, \dots, \lambda_p$ for which $\mathfrak{R}_j = \mathfrak{R} \cap \mathfrak{S}_{\lambda_j} \neq (0)$; then $\mathfrak{R} = \sum_{j=1}^q \dot{+} \mathfrak{R}_j$, and

1. *The functionals $\lambda_1, \dots, \lambda_q$ do not depend on the choice of the subspace $\mathfrak{P}$, for which $\mathfrak{R}$ has maximal dimension.*

These characters $\lambda_1, \dots, \lambda_q$ are called the **special** characters of the representation $x \rightarrow A_x$.

In each $\mathfrak{R}_j$ one can choose a basis $\xi_{j1}, \dots, \xi_{jr_j}$, $r_j = \dim \mathfrak{R}_j$, so that

$$A_x \xi_{jl} = \sum_{s=1}^l \lambda_{jls}(x) \xi_{js}, \quad \lambda_{jll}(x) = \lambda_j(x). \quad (2)$$

Formulas (2) define, for each $j = 1, \dots, q$, a representation of the algebra by triangular matrices with $\lambda_j(x)$ on the diagonal.

Since Π is a finite-dimensional Hilbert space, it follows that

$$\Pi = \sum_{\nu=1}^p \oplus \Pi^\nu, \quad (3)$$

where $\Pi^\nu = \{\xi : \xi \in \Pi, A_x \xi = \lambda_\nu(x) 1 + n \text{ for some } n \in \mathfrak{R}\}$ (some of the Π^ν may be equal to (0)).

Let $\{\eta_{jl}, l = 1, \dots, r_j, j = 1, \dots, q\}$ be a basis in $\mathfrak{R}'$, biorthogonal to $\{\xi_{jl}\}$. Using (1), Theorem 1, and repeating the arguments in (5) (see also (4)), we conclude that the following holds.

Theorem 3. *Every representation $x \rightarrow A_x$ in a separable space Π_k of an algebra R with only real characters is given, in accordance with the decompositions (1), (3) and the bases $\{\xi_{jl}\}, \{\eta_{jl}\}$, by the formulas (2) and*

$$\mathfrak{H} = \int_T \mathfrak{H}(t) d\sigma,$$

$$A_x \{h(t)\} = - \sum_{j=1}^q \sum_{l=1}^{r_j} \left[\int_T (h(t), h_{jl}(x, t)) d\sigma \right] \xi_{jl} + \{x(t)h(t)\}$$

for $\{h(t)\} \in \mathfrak{H}$,

$$A_x \pi^\nu = \sum_{j=1}^q \sum_{l=1}^{r_j} (\pi^\nu, \pi_{jl}^\nu) \xi_{jl} + \lambda_\nu(x) \pi^\nu \quad \text{for } \pi^\nu \in \Pi^\nu, \nu = 1, \dots, p,$$

$$A_x \eta_{jl} = \sum_{\mu=1}^q \sum_{\nu=1}^{r_\mu} a_{j\mu\nu}(x) \xi_{\mu\nu} + \sum_{\nu=1}^{r_j} \overline{\lambda_{j\nu l}(x^*)} \eta_{j\nu} + \{h_{jl}(x^*, t)\} + \sum_{\nu=1}^p \pi_{jl}^\nu(x^*),$$

where $\lambda_1, \dots, \lambda_q$ are special functionals of the representation; T is the space of symmetric maximal ideals of the algebra R ; σ is a Borel measure on T ; $\mathfrak{H}(t)$ is a

σ -measurable family of separable Hilbert spaces; $\lambda_{jls}(x), \alpha_{jl\mu\nu}(x)$ are numerical, and $\{h_{jl}(x, t)\}, \pi_{jl}^\nu(x)$ are vector-functions on $\overline{R}$ with values in $\mathfrak{H}, \Pi^\nu$, continuous on R , where $\alpha_{jl\mu\nu}(x^*) = \alpha_{\mu\nu jl}(\overline{x}), \lambda_j(x^*) = \overline{\lambda_j(x)}$,

$$h_{jl}(x, t) = (\overline{x(t)} - \overline{\lambda_j(x)})\xi_{jl}(t) - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(x)} \xi_{j\mu}(t), \quad j, l = 1, \dots, q,$$

σ -almost everywhere for $t \neq \lambda_j, j = 1, \dots, q$;

$$\pi_{jl}^\nu(x) = [\overline{\lambda_\nu(x)} - \overline{\lambda_j(x)}]\pi_{jl}^\nu - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(x)} \pi_{j\mu}^\nu$$

for $j \neq \nu$; $\{\xi_{j\mu}(t)\}$ is a σ -measurable vector-function on T with values $\xi_{j\mu}(t) \in \mathfrak{H}(t)$; π_{jl}^ν are vectors from Π^ν .

4. Let G be a locally bicomact commutative group with a countable base of neighborhoods, and let $g \rightarrow U_g$ be a unitary representation of the group G in Π_k . Put $\omega(g) = |U_g|$. Let $L_\omega^1(G)$ be the totality of all such measurable functions $x(g)$ on G that $|x| = \int |x(g)|\omega(g) dg < \infty$. From the relations $\omega(g_1 g_2) \leq \omega(g_1)\omega(g_2)$, $\omega(g^{-1}) = |U_{g^{-1}}| = |U_g^*| = |U_g| = \omega(g)$, it easily follows that $L_\omega^1(G)$ is a symmetric algebra, with the usual definition of addition and multiplication by a scalar, with convolution multiplication and involution $x^*(g) = \overline{x(g^{-1})}$. Since G has a countable base of neighborhoods, $L_\omega^1(G)$ is separable.

Let R_ω be the algebra obtained from $L_\omega^1(G)$ by adjoining an identity. Then the results of §§ 2, 3 are applicable to R_ω . Applying the usual arguments, we conclude that the maximal ideals t in R_ω distinct from $t_0 = L_\omega^1(G)$ are given by the relations

$$x^\wedge(t) = \int x(g)\chi(g) dg, \quad x^\wedge(t_0) = 0, \quad (4)$$

where $\chi(g)$ is a continuous function on G satisfying the conditions $\chi(e) = 1$ (e is the identity of the group G), $\chi(g_1 g_2) = \chi(g_1)\chi(g_2)$, $|\chi(g)| \leq C\omega(g)$, where C is some constant; moreover, the correspondence $t \rightarrow \chi$ is thus established one-to-one. The ideal $t \rightarrow \chi$ is symmetric if and only if $\chi(g^{-1}) = \overline{\chi(g)}$, i.e. $|\chi(g)| = 1$, i.e. χ is a character of the group G . Hence it easily follows that the space T of symmetric maximal ideals of the algebra R_ω is homeomorphic to the space G_∞^* , the group G^* of characters of the group G with an infinite point adjoined. We shall therefore identify T with G_∞^* . To the representation $g \rightarrow U_g$ of the group G there corresponds the representation $y \rightarrow A_y$ of the algebra R_ω by the formula $A_y = \lambda 1 + \int x(g)U_g dg$ for $y = \lambda e + x$, $x \in L_\omega^1(G)$. Special f., special s.f. and the

hyperbolic subspace H of the representation $y \rightarrow A_y$ will respectively be called the s.f., special s.f. and hyperbolic subspace of the representation $g \rightarrow U_g$.

II. If $H \neq (0)$, the assertion of Theorem 2 is valid for a unitary representation of the group G in Π_k .

Considering the representation $g \rightarrow U_g$ in $H^\perp$ instead of Π_k , we may henceforth assume that this representation has only real s.f. $\lambda_1, \dots, \lambda_p$; in view of the identification established above of T with G_∞^* , they are characters $\lambda_j = \chi_j \in G^*$, $j = 1, \dots, p$. In particular, the special s.f. $\lambda_j = \chi_j$, $j = 1, \dots, q$, of the representation are called its special characters. It is not hard to verify that

$$A_x^{(1)} = \int x(g) U_g^{(1)} dg, \quad \lambda_{jls}(x) = \int x(g) \lambda_{jls}(g) dg,$$

etc. for $x(g) \in L_\omega^1(G)$, where $g \rightarrow U_g^{j(1)}$ is an ordinary continuous unitary representation of the group G in $\mathfrak{H}$, and $\lambda_{jls}(g)$ is a continuous numerical function on G . Taking into account (4) and these formulas and applying Theorem 3, we arrive at the following result:

Theorem 4. Let G be a commutative locally bicomact group with a countable base of neighborhoods. Every continuous unitary representation $g \rightarrow U_g$ of the group G in a separable space Π_k with only real c.f. is given, in accordance with the decompositions (1) and (3) and the bases $\{\xi_{jl}\}, \{\eta_{jl}\}$, by the formulas

$$\mathfrak{H} = \int_{G^*} \mathfrak{H}(\chi) d\sigma,$$

$$U_g \xi_{jl} = \sum_{s=1}^l \lambda_{jls}(g) \xi_{js}, \quad \lambda_{jll}(g) = \chi_j(g), \quad j = 1, \dots, q; \quad l = 1, \dots, r_q,$$

$$U_g \{h(\chi)\} = - \sum_{j=1}^q \sum_{l=1}^{r_j} \left[\int_G (h(\chi), h_{jl}(g, \chi)) d\sigma \right] \xi_{jl} + \{\chi(g) h(\chi)\},$$

$$U_g \pi^\nu = \sum_{j=1}^q \sum_{l=1}^{r_j} (\pi^\nu, \pi_{jl}^\nu(g)) \xi_{jl} + \chi_\nu(g) \pi^\nu \quad \text{for } \pi^\nu \in \Pi^\nu, \quad \nu = 1, \dots, p,$$

$$U_g \eta_{jl} = \sum_{\mu=1}^q \sum_{\nu=1}^{r_\mu} \alpha_{jl\mu\nu}(g) \xi_{\mu\nu} + \sum_{\nu=1}^{r_j} \overline{\lambda_{j\nu l}(g^{-1})} \eta_{j\nu} + \{h_{jl}(g^{-1}, t)\} + \sum_{\nu=1}^p \pi_{jl}^\nu(g^{-1}),$$

where $\chi_1, \dots, \chi_q$ are distinguished characters of the representation; σ is a Borel measure on G^* ; $\mathfrak{H}(\chi)$ is a measurable family of separable Hilbert spaces;

$\lambda_{jl\mu}(g), \alpha_{jl\mu\nu}(g)$ are numerical functions, and $h_{jl}(g, t), \pi_{jl}^\nu(g)$ are vector-functions on G with values in $\mathfrak{H}$ and Π^ν , continuous on G , with

$$\alpha_{jl\mu\nu}(g^{-1}) = \overline{\alpha_{\mu\nu jl}(g)};$$

$$h_{jl}(g, t) = (\overline{\chi(g)} - \overline{\chi_j(g)})\xi_{jl}(t) - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(g)}\xi_{j\mu}(t), \quad j, l = 1, \dots, q,$$

σ -almost everywhere for $\chi \neq \chi_j, j = 1, \dots, q$;

$$\pi_{jl}^\nu(g) = [\overline{\chi_\nu(g)} - \overline{\chi_j(g)}]\pi_{jl}^\nu - \sum_{\mu=l+1}^{r_j} \overline{\lambda_{j\mu l}(g)}\pi_{j\mu}^\nu, \quad j, l = 1, \dots, q,$$

for $j \neq \nu$; $\{\xi_{j\mu}(\chi)\}$ is a σ -measurable vector-function on G^* with values $\xi_{j\mu}(\chi) \in \mathfrak{H}(\chi)$; π_{jl}^ν are vectors from Π_ν .

For the cases in which G is the additive group of the integers or the additive group of all real numbers, Theorem 4 apparently yields the known results of I. S. Iokhvidov and M. G. Krein (see, for example, ⁽⁶⁾); for the case in which G is the additive n -dimensional group, close results were obtained in an unpublished work of A. I. Shtern.

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