

SEVERAL REMARKS ON THE METHOD OF MULTIPLIERS

Ya. L. Geronimus

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Abstract

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MATHEMATICS

Ya. L. Geronimus

SEVERAL REMARKS ON THE METHOD OF MULTIPLIERS

(Presented by Academician S. N. Bernstein, 9 VI 1965)

I. Let $\Lambda = \{\lambda_k^{(n)}\}$ ($k = 0, 1, \dots, n; n = 1, 2, \dots$) be a triangular matrix of real numbers, with

$$\Delta_2 \lambda_k^{(n)} = \lambda_k^{(n)} - 2\lambda_{k+1}^{(n)} + \lambda_{k+2}^{(n)} \quad (k = 0, 1, \dots, n-1); \quad \lambda_0^{(n)} = 1, \lambda_{n+1}^{(n)} = 0. \quad (1)$$

In the theory of linear methods of summation of Fourier series, an important role is played by estimates for the mean value of the kernel, i.e. for the integral

$$\frac{1}{\pi} \int_0^\pi |K_n(t)| dt, \quad K_n(t) = \frac{\lambda_0^{(n)}}{2} + \sum_{k=1}^n \lambda_k^{(n)} \cos kt. \quad (2)$$

A simple upper estimate is given by

Theorem 1. *If for each value of n the sequence of numbers $\{\lambda_k^{(n)}\}_0^n$ is convex or concave, i.e.*

$$\varepsilon_n \Delta_2 \lambda_k^{(n)} \geq 0 \quad (k = 0, 1, \dots, n-1; n = 1, 2, \dots), \quad \varepsilon_n = \pm 1, \quad (3)$$

then the estimate holds

$$\frac{1}{\pi} \int_0^\pi |K_n(t)| dt \leq \frac{1}{2} + C(n+1)|\lambda_n^{(n)}|; \quad (4)$$

in the particular case where $\varepsilon_n = 1$, $\lambda_n^{(n)} \geq 0$, one may take $C = 0$.

For the proof we apply Abel's transformation twice to the series $K_n(t)$; we obtain

$$K_n(t) = \sum_{k=0}^{n-1} \Delta_2 \lambda_k^{(n)} \cdot S_k(t) + \lambda_n^{(n)} S_n(t), \quad S_k(t) = \frac{1}{2} \left(\frac{\sin \frac{k+1}{2} t}{\sin \frac{1}{2} t} \right)^2,$$

$$\frac{1}{\pi} \int_0^\pi |K_n(t)| dt \leq \frac{1}{2} \sum_{k=0}^{n-1} (k+1) |\Delta_2 \lambda_k^{(n)}| + \frac{1}{2} (n+1) |\lambda_n^{(n)}|;$$

from the obvious equality

$$\lambda_0^{(n)} = \sum_{k=0}^{n-1} (k+1) \Delta_2 \lambda_k^{(n)} + (n+1) \lambda_n^{(n)}$$

we obtain

$$\frac{1}{\pi} \int_0^\pi |K_n(t)| dt \leq \frac{\varepsilon_n \lambda_0^{(n)}}{2} + \frac{n+1}{2} \{ |\lambda_n^{(n)}| - \varepsilon_n \lambda_n^{(n)} \},$$

whence (4) follows.

* L. Fejér ⁽⁵⁾ showed that in this case the kernel is nonnegative.

II. From Theorem 1 it follows: under condition (3), the additional condition

$$|\lambda_n^{(n)}| = O(1/n) \tag{5}$$

is sufficient for the validity of the inequality

$$\frac{1}{\pi} \int_0^\pi |K_n(t)| dt \leq C_1, \tag{6}$$

where C_1 does not depend on n .

Although the simple condition (5) is not necessary, it is nevertheless applicable in almost all known particular cases.

1) The Cesàro method (C, r) , $r \geq 0$; in this case we have

$$\lambda_n^{(n)} = n! \Gamma(r+1) / \Gamma(n+r+1) \simeq e^r \Gamma(r+1) / n^r,$$

i.e., condition (5) is applicable for $r \geq 1$.

2) The Vallée-Poussin method

$$\lambda_k^{(n)} = \begin{cases} 1 & (k = 0, 1, \dots, n-p), \\ (n-k+1)/(p+1) & (k = n-p+1, \dots, n); \end{cases}$$

we have $\lambda_n^{(n)} = 1/(p+1)$, and (5) is equivalent to the condition $\liminf_{n \rightarrow \infty} p/n > 0$.

- 3) Let $\lambda_k^{(n)} = \varphi(k/(n+1))$ ($k = 0, 1, \dots, n$), where $\varphi(0) = 1$, $\varphi(1) = 0$; the function $\varphi(x)$ has on the interval $[0, 1]$ a piecewise continuous second derivative that does not change sign on this interval; then

$$\lambda_{n+1}^{(n)} - \lambda_n^{(n)} = -\lambda_n^{(n)} = \varphi(1) - \varphi\left(\frac{n}{n+1}\right) = \frac{1}{n+1} \varphi'(z),$$

$$\frac{n}{n+1} < z < 1, \quad |\lambda_n^{(n)}| \leq \frac{|\varphi'(z)|}{n+1},$$

i.e., condition (5) is fulfilled owing to the boundedness of $|\varphi'(x)|$ in a neighborhood of the point $x = 1$. In particular, it is fulfilled in the method of S. N. Bernstein, where $\varphi(x) = \cos \pi x/2$, and also in the case $\varphi(x) = (1-x)^\alpha$, if $\alpha \geq 1$, etc.

III. Let $f(\theta) \in \mathcal{L}(-\pi, \pi)$,

$$f(\theta) \sim \sum_{k=0}^{\infty} (a_k \cos k\theta + b_k \sin k\theta), \quad (7)$$

$$U_n(f, \Lambda; \theta) = \frac{\lambda_0^{(n)} a_0}{2} + \sum_{k=1}^n \lambda_k^{(n)} (a_k \cos k\theta + b_k \sin k\theta).$$

Theorem 2. For the convergence

$$\lim_{n \rightarrow \infty} U_n(f, \Lambda; \theta) = f(\theta) \quad (8)$$

at every Lebesgue point of the function $f(\theta)$, the conditions (3), (5), and the additional condition

$$\lim_{n \rightarrow \infty} \lambda_k^{(n)} = 1 \quad (k = 0, 1, \dots) \quad (9)$$

are sufficient.

The proof follows from the results of S. M. Nikol'skii⁽³⁾.

- IV. S. B. Stechkin⁽⁴⁾ posed the question of finding the best constant in the inequality

$$\left| \sum_{k=0}^n \frac{\lambda_k^{(n)}}{n-k+1} \right| \leq C \int_0^\pi |K_n(t)| dt. \quad (10)$$

Using our results ⁽²⁾ and the results of N. I. Akhiezer and M. G. Krein ⁽¹⁾, one can give an algorithm for solving this problem.

Theorem 3. *The best value of the constant C in (10) is equal to the largest positive root of the equation*

$$\begin{vmatrix} \gamma_0 & \gamma_1 & \cdots & \gamma_n \\ \gamma_{-1} & \gamma_0 & \cdots & \gamma_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ \gamma_{-n} & \gamma_{-n+1} & \cdots & \gamma_0 \end{vmatrix} = 0, \quad \gamma_{-k} = \overline{\gamma_k}, \quad \gamma_0 = \gamma + \overline{\gamma}, \quad (11)$$

where $\gamma, \{\gamma_k\}_1^n$ are the first coefficients of the expansion

$$\gamma + \sum_{k=1}^n \gamma_k z^k + O(z^{n+1}) = \exp \left\{ \frac{i}{C} \sum_{k=0}^n \frac{z^k}{n-k+1} + O(z^{n+1}) \right\}. \quad (12)$$

Kharkov
Aviation Institute

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Note: Figure translations are in progress. See original paper for figures.

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