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Abstract

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PHYSICS

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ENERGY SPECTRUM FOR STOCHASTIC OSCILLATIONS OF THE SURFACE OF A LIQUID

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The problem of the energy spectrum for stochastic gravitational waves on the surface of a liquid (ripples on water) has been studied experimentally ^(1,2). It was established that there exists a range of wave numbers (the “equilibrium range”) in which the spectrum has a universal character and does not depend on the method of excitation of the oscillations or on the magnitude of the viscosity. The experimental data show that the spectrum in the equilibrium range has the form $E_\omega d\omega \sim \omega^{-s} d\omega$, $s = 5 \div 6$. Phillips ⁽³⁾ put forward the hypothesis that $E^\omega \sim \omega^{-5}$ (on the basis of dimensional analysis) and gave phenomenological arguments in favor of this hypothesis.

In the present work the question of the spectrum of stochastic oscillations was studied with the aid of equations for the correlation functions of the displacement of the surface of the liquid from equilibrium. It is shown that the exact solution for a homogeneous, isotropic, stationary statistical distribution is $E_\omega \sim \omega^{-4}$. The method for obtaining such solutions was developed earlier by one of the authors for certain model problems ⁽⁴⁾.

In the “equilibrium range,” where the influence of viscosity is small, the main role is played by the nonlinear interaction between waves. If the nonlinearity is not too large, then $\tilde{\gamma}$ —the damping of waves due to nonlinear interaction—is small in comparison with the frequency ω , so that one may use an expansion in $\tilde{\gamma}/\omega^*$ (this method is usually applied in the study of weak plasma turbulence) ^(5,6).

Let us choose as the variables describing the oscillations of the liquid: $\Phi(x, y, z, t)$ —the velocity potential; $\Psi(x, y, z, t) \equiv \Phi(x, y, z, t)|_{z=\eta}$; $\eta(x, y, z, t)$ —the displacement of the surface from its equilibrium value. Then the system of equations describing the oscillations of the liquid takes the form

$$\frac{\partial \eta}{\partial t} - A = -\frac{\partial \eta}{\partial x} \frac{\partial \Psi}{\partial x} - \frac{\partial \eta}{\partial y} \frac{\partial \Psi}{\partial y} + A \left[\left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right],$$

$$\frac{\partial \Psi}{\partial t} + g\eta = -\frac{1}{2} \left[\left(\frac{\partial \Psi}{\partial x} \right)^2 + \left(\frac{\partial \Psi}{\partial y} \right)^2 \right] + \frac{A^2}{2} + \frac{A^2}{2} \left[\left(\frac{\partial \eta}{\partial x} \right)^2 + \left(\frac{\partial \eta}{\partial y} \right)^2 \right], \quad (1)$$

$$\Delta \Phi = 0,$$

where $A \equiv \partial \Phi / \partial z|_{z=\eta}$, with $\rho = 1$.

We shall assume that the depth of the liquid is infinite. Then

$$\Phi(x, y, z, t) = \int_{-\infty}^{\infty} \Phi_{\mathbf{k}} e^{|\mathbf{k}|z} e^{i\mathbf{k}\mathbf{r}} d\mathbf{k}, \quad \Psi(x, y, t) = \sum_{n=0}^{\infty} \frac{1}{n!} \eta^n(x, y) e^{i\mathbf{k}\mathbf{r}} d\mathbf{k},$$

where $\mathbf{k} = (k_x, k_y)$ and $\mathbf{r} = (x, y)$ are two-dimensional vectors.

* In contrast to ordinary hydrodynamic turbulence, this permits a rigorously justified splitting of the chains of equations for the correlation functions.

Let us pass in equations (1) to the Fourier transform with respect to the coordinates and express the Fourier component $A_{\mathbf{k}}$ in terms of the Fourier components Ψ_k up to cubic terms:

$$\begin{aligned} A_k = & |\mathbf{k}| \Psi_k + \int |\mathbf{k}_1| (|\mathbf{k}_1| - |\mathbf{k}|) \Psi_{k_1} \eta_{k_2} \delta_{k-k_1-k_2} dk_1 dk_2 + \\ & + \int |\mathbf{k}_1| \left[\frac{|\mathbf{k}_1| (|\mathbf{k}_1| - |\mathbf{k}|)}{2} + |\mathbf{k} - \mathbf{k}_2| (|\mathbf{k}| - |\mathbf{k} - \mathbf{k}_2|) \right] \Psi_{k_1} \eta_{k_2} \eta_{k_3} \times \\ & \times \delta_{k-k_1-k_2-k_3} dk_1 dk_2 dk_3. \end{aligned} \quad (2)$$

In addition, let us introduce the change of variables

$$\eta_k = (|\mathbf{k}|/g)^{1/4} (a_k + a_{-k}^*), \quad \Psi_k = -i \left(\frac{g}{|\mathbf{k}|} \right)^{1/4} (a_k - a_{-k}^*).$$

Then the equations take the form

$$\begin{aligned} \frac{\partial a_k}{\partial t} + i\omega_k a_k = & -i \int \left[V_{kk_1 k_2}^{(1)} a_{k_1} a_{k_2} \delta_{k-k_1-k_2} + 2V_{k_2 k_1 k}^{(1)} a_{k_1}^* a_{k_2} \delta_{k+k_1-k_2} + \right. \\ & \left. + V_{kk_1 k_2}^{(2)} a_{k_1}^* a_{k_2}^* \delta_{k+k_1+k_2} \right] dk_1 dk_2 - i \int W_{kk_1 k_2 k_3} a_{k_1}^* a_{k_2} a_{k_3} \delta_{k+k_1-k_2-k_3} dk_1 dk_2 dk_3, \\ \omega_k = & \sqrt{|\mathbf{k}|g}. \end{aligned} \quad (3)$$

Fig. 1

Figure 1: Fig. 1

The remaining cubic terms may be neglected. Here $V_{kk_1k_2}^{(1)}$ and $V_{kk_1k_2}^{(2)}$ are homogeneous functions of degree 7/4, satisfying the symmetry relations

$$V_{kk_1k_2}^{(1)} = V_{kk_2k_1}^{(1)}, \quad V_{kk_1k_2}^{(2)} = V_{k_1kk_2}^{(2)} = V_{kk_2k_1}^{(2)} \quad (4)$$

and $W_{kk_1k_2k_3}$ is a homogeneous function of degree 3, satisfying the symmetry conditions

$$W_{kk_1k_2k_3} = W_{k_1kk_2k_3} = W_{kk_1k_3k_2} = W_{k_2k_3kk_1}. \quad (5)$$

For a statistical description of the oscillations we shall use Wyld's diagram technique⁽⁷⁾. Since the equations contain not only the variables a_k , but also a_k^* , all lines in the graphs will be oriented. We shall assume the vertex functions to be unperturbed, and for the one-particle correlation function $N_{k\omega} = \langle a_{k\omega} a_{k\omega}^* \rangle$ (wavy line) and the propagation function (bold straight line) one obtains a system of equations (Fig. 1). A thin straight line corresponds to the unperturbed Green's function

$$\frac{1}{\omega + i\gamma - \omega_k}.$$

Each graph is the sum of all possible graphs of the same topology, but with differently oriented lines. All subsequent graphs are of higher order in γ/ω .

Fig. 1

Next, let us substitute in the equations $N_{k\omega} = N_k \delta(\omega - \omega_k) + \Phi_{k\omega}$ and express $\Phi_{k\omega}$ in terms of N_k up to quartic terms. Then we integrate the equation with respect to ω . We obtain

$$\begin{aligned} & \int U_{kk_1k_2k_3} (N_{k_1} N_{k_2} N_{k_3} + N_k N_{k_2} N_{k_3} - N_k N_{k_1} N_{k_2} - \\ & - N_k N_{k_1} N_{k_3}) \delta_{k+k_1-k_2-k_3} \delta_{\omega_k+\omega_{k_1}-\omega_{k_2}-\omega_{k_3}} dk_1 dk_2 dk_3 = 0. \end{aligned} \quad (6)$$

Here

$$U = \left| \frac{2V_{k+k_1,k,k_1} V_{k+k_1,k_2,k_3}}{\omega_{k+k_1} - \omega_k - \omega_{k_1}} + \frac{U_{-k-k_1,k,k_1} U_{-k-k_1,k_2,k_3}}{\omega_{k-k_1} - \omega_k + \omega_{k_1}} + \right.$$

Fig. 2

Figure 2: Fig. 2

$$\begin{aligned}
& + 2 \frac{V_{k,k_1,k-k_1} V_{k_2,k-k_1,k_2}}{\omega_{k-k_1} - \omega_k + \omega_{k_1}} + 2 \frac{V_{k,k_1,k-k_1} V_{k_3,k-k_1,k_2}}{\omega_{k-k_1} - \omega_k + \omega_{k_1}} + 2 \frac{V_{k_1,k,k-k_1} V_{k_2,k_3,k_1-k}}{\omega_{k_1-k} + \omega_k - \omega_{k_1}} + \\
& + 2 \frac{V_{k_1,k,k-k_1} V_{k_3,k_2,k_1-k}}{\omega_{k_1-k} + \omega_k - \omega_{k_1}} + W_{kk_1k_2k_3} \Big|^2.
\end{aligned}$$

U is a homogeneous function satisfying the same symmetry conditions as W . Averaging equation (6) over the angles in k -space and passing to the variables $\omega = \sqrt{kg}$, we obtain

$$\begin{aligned}
& \int T_{\omega, \omega_1 + \omega_2 - \omega, \omega_1, \omega_2} \{ N_{\omega} N_{\omega_2} N_{\omega_1 + \omega_2 - \omega} + N_{\omega} N_{\omega_1} N_{\omega_2} - \\
& - 2 N_{\omega} N_{\omega_1} N_{\omega_1 + \omega_2 - \omega} \} d\omega_1 d\omega_2 = 0.
\end{aligned} \tag{7}$$

Fig. 2

The integration is carried out over the region shown in Fig. 2, where the curves 1, 2, 3, 4 are described respectively by the equations:

1. $(\omega_1 + \omega_2 - \omega)^2 = -\omega_1^2 - \omega_2^2 + \omega^2$.
2. $(\omega_1 + \omega_2 - \omega)^2 = \omega_1^2 + \omega_2^2 + \omega^2$.
3. $(\omega_1 + \omega_2 - \omega)^2 = -\omega_1^2 + \omega_2^2 - \omega^2$.
4. $(\omega_1 + \omega_2 - \omega)^2 = \omega_1^2 - \omega_2^2 - \omega^2$.

The function T is positive, homogeneous of degree 20, and satisfies the symmetry conditions (5). We shall seek solutions of equation (7) in the form $N_{\omega} = \omega^s$. Let us divide the integration region into the regions I, II, III, IV and map each of the regions II, III, IV onto region I according to the following rules:

- 1) For region II

$$\omega_1 \rightarrow \frac{\omega_2 \omega}{\omega_1 + \omega_2 - \omega}; \quad \omega_2 \rightarrow \frac{\omega_1 \omega}{\omega_1 + \omega_2 - \omega}.$$

- 2) For region III

$$\omega_2 \rightarrow \frac{\omega^2}{\omega_2}; \quad \omega_1 \rightarrow \frac{(\omega_1 + \omega_2 - \omega)\omega}{\omega_2}.$$

3) For region *IV*

$$\omega_1 \rightarrow \frac{\omega^2}{\omega_1}; \quad \omega_2 \rightarrow \frac{(\omega_1 + \omega_2 - \omega)\omega}{\omega_1}.$$

Using the symmetry and homogeneity of T , we obtain

$$\begin{aligned} & \int_I \frac{T_{\omega, \omega_1 + \omega_2 - \omega, \omega_1, \omega_2}}{(\omega_1 + \omega_2 - \omega)^{23+3s} \omega_1^{23+3s} \omega_2^{23+3s}} [\omega_1^s \omega_2^s (\omega_1 + \omega_2 - \omega)^s + \omega^s \omega_1^s \omega_2^s - \\ & - 2\omega^s \omega_1^s (\omega_1 + \omega - \omega)^s] [\omega_1^{23+3s} \omega_2^{23+3s} (\omega_1 + \omega_2 - \omega)^{23+3s} + \\ & + \omega^{23+3s} \omega_1^{23+3s} \omega_2^{23+3s} - 2\omega^{23+3s} \omega_2^{23+3s} (\omega_1 + \omega_2 - \omega)^{23+3s}] d\omega_1 d\omega_2 = 0. \end{aligned}$$

The integrand vanishes for $s = -1$ and $s = -8$. However, when these solutions are substituted into equation (7), it is found that for $s = -1$ the integral diverges at large k , whereas for $s = -8$ the integral converges both at large and at small k .

The first solution is the Rayleigh-Jeans distribution. In our case it has no physical meaning. The second solution (to which the energy spectrum $E_\omega = \omega^4 N_\omega = A\omega^{-4}$ corresponds) is an exact analogue of the Kolmogorov spectrum. In the real problem, with a source at small k and viscosity at large k , it describes the energy flux into the dissipation region. As analysis shows, the magnitude of this flux P does not depend on the viscosity coefficient, and moreover $P \sim A^3$.

The difference between the spectrum obtained by us and the Phillips spectrum ($E_\omega \sim \omega^{-5}$) is explained by the fact that Phillips considered wave motion with the formation of whitecaps, which corresponds to a strong nonlinear interaction of waves. In this case the expansion in the parameter γ/ω becomes inapplicable.

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