

ON THE CONSTRUCTION OF CUBATURE FORMULAS WITH A REGULAR BOUNDARY LAYER

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Abstract

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MATHEMATICS

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ON THE CONSTRUCTION OF CUBATURE FORMULAS WITH A REGULAR BOUNDARY LAYER

In the author's note ⁽¹⁾, a convenient method was published for the direct determination of the coefficients of the boundary layer of cubature formulas, based on the Fourier transform, and a calculation was given of the cubature layer for certain 3-dimensional formulas. In the n -dimensional case, when the geometric clarity of all the constructions is lost, certain additional questions of a logical and combinatorial nature arise, on whose solution success in the construction of such formulas depends. Here we shall elucidate these questions.

The boundary layer of a cubature formula arises (see ^(2, 3)) in the construction of the error functional by adding elementary functionals according to the formula

$$l(x) = \sum_{h^{-1}H^{-1}\gamma \in \Omega} l_0(x - h^{-1}H^{-1}\gamma) + \sum_{\gamma \in B_1^{\gamma}} l(x - h^{-1}H^{-1}\gamma). \quad (1)$$

We shall consider convex polyhedra and an integral lattice R . Since an affine transformation preserves all properties of the boundary layer, the latter restriction does not affect generality. Suppose that all k -dimensional faces of the domain Ω under consideration are rational and each contains a k -dimensional sublattice R_k of the main lattice R . Continuing each such face $\Gamma_k^{(j)}$ without restriction together with all faces of higher dimension meeting it, we obtain a certain angle $\Omega_k^{(j)}$ with a k -dimensional vertex. This angle remains invariant under any of its shifts by a vector of the lattice R_k .

In formula (1), among the terms $l_{\gamma}(x - h^{-1}\gamma)$, whose support contains no other boundary points except $\Gamma_k^{(j)}$ and the faces of higher dimension $\Gamma_{k+l}^{(j)}$ intersecting it, we combine into one class all those for which the difference of the indices $\gamma^{(1)} - \gamma^{(2)}$ belongs to R_k . In constructing the boundary layer we shall take the corresponding summands to be equal for the whole class. In this case, at the lattice points lying in a neighborhood of such an angle and differing by a vector $\gamma \in R_k$, the coefficients of the cubature formula will be equal. We arrive at the conclusion:

Theorem 1. *For every rational polyhedron there exist cubature formulas of the form*

$$l(x) = \varepsilon_{\Omega}(x) \left[1 - \Phi_0(h^{-1}x) - \sum_{k,j} \psi_k^{(j)}(x) \right], \quad (2)$$

where by $\psi_k^{(j)}(x)$ is denoted the point functional

$$\psi_k^{(j)}(x) = \sum_{h^{-1}\gamma \in \Omega_k^{(j)}} C_{\gamma} \delta(x - h^{-1}\gamma), \quad (3)$$

and C_{γ} in formula (3) is invariant under shifts by a vector $\gamma \in R_k$.

In note ⁽¹⁾, error functionals of order m were introduced for infinite domains, whose Fourier transform has a zero of order-

m at the origin. These functionals make it possible to construct cubature formulas for conditionally finite functions, i.e., those for which the intersection of the support with the given angle $\Omega_k^{(j)}$ is bounded.

Theorem 2. *Every functional of order m of the form (2) for a convex $(n-k)$ -hedral angle with a k -dimensional face at the vertex admits a representation*

$$l(x) = \sum_{\gamma \geq 0} l_0(x - h^{-1}\gamma), \quad (4)$$

where $l_0(x)$ is orthogonal to x^{α} ($|\alpha| = m$), i.e., is a functional with a regular boundary layer.

We shall carry out the proof for the case when the angle has a zero-dimensional vertex, its faces are the coordinate axes $x_1, x_2, \dots, x_n$, and the lattice is the union of several cubic integer lattices shifted with respect to one another by a vector whose components are proper rational fractions. The general case is reduced to this one by an affine transformation.

The functional $l_0(x)$ in formula (4) for this case may be taken in the form

$$l_0(x) = \sum (-1)^{n+\dim \rho} l(x - x^{(\rho)}), \quad (5)$$

where ρ denotes, as in (4), a proper n -digit binary fraction, and the vector $x^{(\rho)}$ has components respectively equal to the digits of the binary places of ρ ; $\dim \rho$ is the total number of ones occurring in these places.

The validity of (4) is established by a direct verification.

Theorem 3. *Every error functional of order m of the form (4) for a convex angle with a k -dimensional face will be a functional with a regular boundary layer.*

For an $(n-k)$ -hedral angle this theorem is just Theorem 2 proved above. In the general case we divide the angle into a finite number of $(n-k)$ -hedral angles, which corresponds to a decomposition into simplices of the figure obtained by intersecting this angle with an $(n-k-1)$ -dimensional plane. Adding to each such simplest functional mixed boundary layers, consisting of exterior and interior points, odd with respect to the boundary, we transform each of the resulting functionals again into a functional of order m . The sum of functionals of order m is again a functional of order m . The theorem is proved.

Theorem 4. *The error functional $l_0(x)$ of order m of the form (4) for any bounded convex polyhedron is represented in the form of a linear combination of the error functionals of all its convex k -hedral angles:*

$$l_\Omega(x) = - \sum_{s=0}^n (-1)^{n-s} \sum_{j=1}^{Q(s)} l_{\Omega_s^{(j)}}(x). \quad (6)$$

Let us first prove the analogous formula for the characteristic functions of polyhedra,

$$\mathcal{E}_\Omega(x) + \sum_{s=0}^n (-1)^{n-s} \sum_{j=1}^{Q(s)} \mathcal{E}_{\Omega_s^{(j)}}(x) = 0. \quad (7)$$

Formula (7) for interior points, where each summand is equal to one, coincides with Euler's formula for the alternating sum of the numbers of faces of different dimensions of a convex polyhedron of genus zero.

For exterior points, as is easy to verify, in the sum on the left only those summands that correspond to faces visible from the interior of the angle will be different from zero. These faces form an open unbounded polyhedron, for which, by the same Euler theorem, the corresponding alternating sum is equal to zero (taking into account the summand $s = n$), as was required to prove.

Returning to (2) and replacing $\mathcal{E}_\Omega(x)$ in $(1 - \Phi_0(h^{-1}x))\mathcal{E}_\Omega(x)$ by its expression from (7), we obtain formula (6).

Let us call a sheet of the boundary layer $\psi_k^{(j)}$ a function of the form

$$C_k^{(j,k)} \sum \delta(x - \gamma - \gamma_k^{(j,s)}), \quad (8)$$

where γ runs through all possible values from the lattice R_k . A portion of the boundary layer consists of portions of each sheet inside some convex unbounded

polyhedron in the corresponding linear space. Representing the characteristic function of such a polyhedron in terms of the characteristic functions of its angles with vertices of dimensions $l < k$, in the same way as was done in Theorem 4, and replacing each such angle by a sum of $(k - l)$ -faces, we can express any portion of a sheet $\omega^{(j,s)}$ of the boundary layer, belonging to $\psi_k^{(j)}$, in the form

$$\omega_{k,l}^{(j,s)} = \quad (9)$$

$$= C_{k,l}^{(j,s)} \sum_{r=1}^Q \left\{ \prod_{s=1}^{q_1} \delta(a^{(s)}x + \gamma_r^{(s)}) \prod_{s=q_1+1}^{q_2} \Phi_0(a^{(s)}x + \gamma_r^{(s)}) \prod_{s=q_2+1}^n \Phi_1(a^{(s)}x + \gamma_r^{(s)}) \right\}.$$

Here, of course, the numbers Q, q_1, q_2 and the constants $a^{(s)}$ and $\gamma_r^{(s)}$ depend on k, j , and l .

Recall that

$$\Phi_0(x) = \sum_{k=-\infty}^{+\infty} \delta(x - k), \quad \Phi_1(x) = \frac{1}{2} \delta(x) + \sum_{k=1}^{\infty} \delta(x - k). \quad (10)$$

The Fourier transforms of the functions $\Phi_0(x), \Phi_1(x)$, and $\delta(x)$ are elementary (see (5)):

$$\tilde{\Phi}_0(p) = \Phi_0(p); \quad \tilde{\Phi}_1(p) = \frac{1}{2} \Phi_0(p) + \frac{1}{2} \operatorname{ctg} \pi p, \quad \tilde{\delta}(p) = 1, \quad (11)$$

where

$$\tilde{f}(p) = \int e^{2\pi i p x} f(x) dx. \quad (12)$$

Using this, we can obtain the coefficients of the boundary layer of any order, beginning with the $(n - 1)$ -dimensional one, and step by step determine these coefficients for layers of lower dimensions.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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