

TOPOLOGICAL EQUIVALENCE

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Abstract

Full Text

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MATHEMATICS

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TOPOLOGICAL EQUIVALENCE OF ALL SEPARABLE BANACH SPACES

(Presented by Academician L. V. Kantorovich, XI 1, 1965)

In this note the following is proved.

Theorem. *All separable Banach spaces are homeomorphic.*

Thus an affirmative answer is obtained to the question posed by M. Fréchet ⁽¹⁾ in 1928 and S. Banach ⁽²⁾ in 1932.

C. Bessaga and A. Pełczyński ⁽³⁾ showed that if a separable Banach space E contains a subspace X homeomorphic to l_2 , then it itself is also homeomorphic to l_2 . It is well known that every infinite-dimensional Banach space contains an infinite-dimensional subspace with a basis. Hence, by the Bessaga-Pełczyński theorem, it is enough to establish a homeomorphism of all Banach spaces with a basis.

Proposition 1. *Let X be a Banach space with basis $\{e_k\}_1^\infty$; denote the conjugate basis system by $\{f_k\}_1^\infty$ ($f_k \in X^*$). Suppose, further, that the norm of the space is subject to the following condition: if*

$$\|x_\nu\| = \|x\| = 1 \quad (\nu = 1, 2, \dots); \quad \lim_{\nu \rightarrow \infty} f_n(x_\nu) = f_n(x) \quad (n = 1, 2, \dots),$$

then

$$\lim_{\nu \rightarrow \infty} \|x_\nu - x\| = 0.$$

Assume also that on the unit ball U of the space X there is defined a functional $F(x)$ having the following properties:

1. *The functional $F(x)$ is continuous.*
2. *$F(x) > 0$ for $\|x\| < 1$; $F(x) = 0$ for $\|x\| = 1$; $F(0) = 1$.*
3. *If*

$$\lim_{n \rightarrow \infty} F(S_n) = 0 \quad \left(S_n = \sum_{k=1}^n a_k e_k \right), \quad (1)$$

then the series $\sum_{k=1}^{\infty} a_k e_k$ converges.

4. For fixed n and a_k ($k < n$), the function

$$\psi(a) = F(S_{n-1} + a e_n)$$

is strictly increasing for $a < 0$ and strictly decreasing for $a > 0$.

Then the space X is homeomorphic to the space l_2 .

We shall precede the proof of Proposition 1 by two lemmas. To each normalized element

$$x = \sum_{k=1}^{\infty} f_k(x) e_k$$

we assign a pair of numerical sequences:

$$\gamma_0(x) = 1; \quad \gamma_n(x) = F(S_{nx}) \quad \left(S_{nx} = \sum_{k=1}^n f_k(x) e_k, \quad n = 1, 2, \dots \right),$$

$$\theta_n(x) = \text{sign } f_n(x) \quad (n = 1, 2, \dots).$$

We note that

$$1 \geq \gamma_1(x) \geq \gamma_2(x) \geq \dots, \quad \lim_{n \rightarrow \infty} \gamma_n(x) = 0.$$

Lemma 1. A normalized sequence x_ν converges to an element x if and only if

$$\lim_{\nu \rightarrow \infty} \theta_n(x_\nu) [\gamma_{n-1}(x_\nu) - \gamma_n(x_\nu)] = \theta_n(x) [\gamma_{n-1}(x) - \gamma_n(x)] \quad (n = 1, 2, \dots).$$

Lemma 2. Whatever the pair of numerical sequences $\{\gamma_k\}_1^\infty$ and $\{\theta_k\}_1^\infty$, subject to the conditions

$$1 \geq \gamma_1 \geq \gamma_2 \geq \dots, \quad \lim_{n \rightarrow \infty} \gamma_n = 0,$$

$$\theta_n = \pm 1, \quad \text{if } \gamma_n \neq \gamma_{n-1}; \quad \theta_n = 0, \quad \text{if } \gamma_n = \gamma_{n-1},$$

there exists a unique normalized element x such that

$$\gamma_n(x) = \gamma_n, \quad \theta_n(x) = \theta_n \quad (n = 1, 2, \dots).$$

Proof of Proposition 1. Let X and Y be spaces satisfying all the requirements of Proposition 1. Suppose that in each of them the functionals $\gamma_n(x), \theta_n(x)$ and, respectively, $\gamma_n(y), \theta_n(y)$ have been introduced. To each normalized element $x \in X$ we assign that normalized element $y = Tx \in Y$ for which

$$\theta_n(y)\gamma_n(y) = \theta_n(x)\gamma_n(x) \quad (n = 1, 2, \dots).$$

By Lemma 2 this correspondence is one-to-one. By Lemma 1 it is bicontinuous. The homeomorphism thus obtained extends to the whole space X :

$$Tx = \|x\|T(x/\|x\|), \quad T(\theta) = \theta.$$

It remains to show that in the space l_2 there exists a functional $F(x)$ with the required properties. It turns out that in this case it is sufficient to put $F(x) = 1 - \|x\|$ ($x \in l_2$).

We now show that every Banach space with a basis can, by means of an equivalent renorming, be made to satisfy the conditions of Proposition 1.

Proposition 2. In every separable Banach space X with basis $\{e_k\}_1^\infty$ one can introduce a norm ($\|\cdot\|$), equivalent to the original one ($\|\cdot\|_0$), satisfying the following requirements:

- a) The space $(X, \|\cdot\|)$ is locally uniformly convex (UR_L in the notation of M. Day, ⁽⁴⁾, p. 188);
- b) For any normalized elements x_ν and x , the condition

$$\lim_{\nu \rightarrow \infty} f_n(x_\nu) = f_n(x) \quad (n = 1, 2, \dots)$$

implies strong convergence:

$$\lim_{\nu \rightarrow \infty} \|x_\nu - x\| = 0.$$

- c) With respect to the new norm the basis $\{e_k\}$ is orthogonal, i.e.

$$\left\| \sum_{k=1}^{n-1} a_k e_k \right\| < \left\| \sum_{k=1}^n a_k e_k \right\|, \quad \text{if } a_n \neq 0 \quad (n = 1, 2, \dots).$$

A norm satisfying the indicated requirements was constructed in ^(5,6) (meaning the norm defined by formula (3) of the note ⁽⁶⁾).

Proposition 3. In a Banach space whose norm satisfies the conditions a), b), c) of Proposition 2, there exists a functional $F(x)$ possessing properties 1-4 formulated in Proposition 1.

Here we shall need two more lemmas. Consider the functional

$$\varepsilon(x, \delta) = \frac{1}{2} \sup_{z \in G(x, \delta)} \|x - z\| \quad (\|x\| = 1, 0 \leq \delta \leq 1),$$

where

$$G(x, \delta) = \{z : \|z\| \leq 1; \min_{0 \leq \lambda \leq 1} \|\lambda x + (1 - \lambda)z\| \geq 1 - \delta\}.$$

If the space is locally uniformly convex, then

$$\lim_{\delta \rightarrow 0} \varepsilon(x, \delta) = 0. \quad (2)$$

Lemma 3. For any Banach space the functional $\varepsilon(x, \delta)$ satisfies the inequalities:

$$\begin{aligned} \delta &\leq \varepsilon(x, \delta) \leq 1, & \varepsilon(x, \delta) &\leq \frac{1}{2} \|x - y\| + \varepsilon(y, \delta + \|x - y\|) \\ \varepsilon(x, \delta + h) - \varepsilon(x, \delta) &\leq 3h/\delta & (0 \leq \delta \leq \delta + h \leq 1), \end{aligned} \quad (3)$$

whence, in particular, it follows that $\varepsilon(x, \delta)$ is uniformly continuous on the set $S \times [\delta_0, 1]$ for every $\delta_0 > 0$ (S is the unit sphere of the space). If the space is locally uniformly convex, then $\varepsilon(x, \delta)$ is continuous on $S \times [0, 1]$.

Let us consider the functional

$$\Phi(x) = \varepsilon\left(\frac{x}{\|x\|}, 1 - \|x\|\right) \quad (0 < \|x\| \leq 1);$$

for $x = \theta$ put $\Phi(\theta) = 1$.

Lemma 4. The functional $\Phi(x)$ is continuous for $\|x\| \leq 1$ and uniformly continuous for $\|x\| \leq 1 - \delta_0$ for every $\delta_0 > 0$.

For each element x ($\|x\| \leq 1$) define the set

$$L(x) = \bigcup_n L_n(x),$$

where $L_n(x)$ is the segment joining $S_{n-1}x$ and S_nx ($n = 1, 2, \dots$); $S_0x = \theta$. Finally, define the functional

$$F(x) = (1 - \frac{1}{2}\|x\|) \inf_{z \in L(x)} \Phi(z) \quad (0 \leq \|x\| \leq 1). \quad (4)$$

Proof of Proposition 3. We have to verify that the functional $F(x)$ has properties 1-4 of Proposition 1. Properties 1 and 2 follow almost immediately from the definition of $F(x)$ and Lemma 4. Let us prove property 3. Suppose (1) is fulfilled. For each S_n consider a point $\sigma_n \in L(S_n)$ at which the lower bound in expression (4) is attained:

$$\sigma_n = S_{m-1} + \alpha e_m \quad (m = m(n), 0 < |\alpha| \leq |a_m|, \alpha a_m > 0). \quad (5)$$

According to (2), (3), and (5), the diameter of the set $G(\sigma_n/\|\sigma_n\|; 1 - \|\sigma_n\|)$ tends to zero as n increases. Consider also the decreasing sequence of closed sets

$$Q(\sigma_n) = \{z : \|z\| \leq 1; f_k(z) = a_k \ (k = 1, 2, \dots, m)\} \quad (n = 1, 2, \dots).$$

Using the orthogonality of the basis $\{e_k\}$, one can establish that

$$G(\sigma_n/\|\sigma_n\|; 1 - \|\sigma_n\|) \supset Q(\sigma_n).$$

Thus the diameter of the set $Q(\sigma_n)$ tends to zero as n increases. The unique element x lying in the intersection of all $Q(\sigma_n)$ will also be the limit of the sequence

$$S_n : x = \sum_{k=1}^{\infty} a_k e_k.$$

Finally, let us prove property 4. Consider the function

$$\psi(\alpha) = F(S_{n-1} + \alpha e_n).$$

Let $|\alpha_1| < |\alpha_2|$, $\alpha_1 \alpha_2 \geq 0$. Then

$$\|S_{n-1} + \alpha_1 e_n\| < \|S_{n-1} + \alpha_2 e_n\| \quad (6)$$

by virtue of the orthogonality of the basis, and

$$L(S_{n-1} + \alpha_1 e_n) \subset L(S_{n-1} + \alpha_2 e_n) \quad (7)$$

by the definition of the set $L(x)$. From (4), (6), and (7) we obtain that $\psi(\alpha_1) > \psi(\alpha_2)$.

The theorem formulated at the beginning of the note is a direct consequence of Propositions 1-3 and the theorem of Bessaga-Pełczyński.

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