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B. P. Kufarev

1966

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Abstract

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UDC 517.54

MATHEMATICS

B. P. KUFAREV

RELATIONS OF THE TYPE OF THE “LENGTH AND AREA PRINCIPLE”

(Presented by Academician M. A. Lavrent'ev, 20 XII 1965)

In the study of conformal, quasiconformal, and more general mappings of plane domains, one inequality is very useful, relating areas and lengths and expressing the so-called “length and area principle.” In the present work we give certain more general relations of this type for mappings having generalized first partial derivatives with respect to x and y and a finite Dirichlet integral. We also indicate one sufficient condition, following from these relations, for a set situated on the boundary of a domain to be mapped into a set of length zero.

Let F be a closed set in Euclidean n -dimensional space R_n , $F \neq R_n$, and let $r(x)$ be the distance from the point $x \in R_n$ to F . Denote by E_t the set $\{x : r(x) = t\}$, the level set of the function $r(x)$.

The proof of the relations (3)–(6) given below is based on the following extension of Fubini's theorem.

Theorem 1. *If a real function $f(x)$ is integrable on R_n and is equal to zero almost everywhere on F , then*

$$\int f \, dm = \int_0^\infty \int_{E_t} f \, d\nu \, dt, \quad (1)$$

where m is Lebesgue measure in R_n , and ν is the $(n-1)$ -dimensional Hausdorff measure (⁽¹⁾, p. 111, or ⁽²⁾, p. 92), defined on subsets of the subspace $E_t \subset R_n$.

The proof of this theorem is carried out by the standard method of the theory of integration, starting from the representation, given in our work ⁽³⁾, of the Lebesgue measure of a measurable set $M \subset R_n$ with measure $m(M \cap F) = 0$:

$$mM = \int_0^\infty \nu(E_t \cap M) \, dt. \quad (2)$$

Let us note that the usual formulation of Fubini's theorem for Euclidean $(k+l)$ -dimensional space is obtained from Theorem 1 by induction on the number k (as the set F one then takes a $(k-1+l)$ -dimensional coordinate plane).

The assertions formulated below concern mappings of plane domains; however, apparently, with corresponding modifications, they can be extended to domains of the space R_n of higher dimension.

Using relation (2), it is not difficult to prove that, for almost all $t \in [0, \infty)$, every simple arc $\Gamma \subset E_t$ is rectifiable.

Theorem 2. *Let G be a domain and F a closed set in the z -plane R_2 . If a real-valued function $f(x, y)$ is continuously differentiable in the domain G , then for almost all $t \in (t_1, t_2)$, $t_1 = \inf_{z \in G} r(z)$, $t_2 =$*

$$= \sup_{z \in G} r(z)$$

the composite function $f[z(t, s)] \equiv f[x(t, s), y(t, s)]$ is absolutely continuous inside $E_t \cap G$, i.e., is absolutely continuous on every simple arc $\Gamma \subset E_t \cap G$ as a function of the arc length s of the curve Γ .

As is known, the derivative of a continuously differentiable function $f(x, y)$ in the direction of the tangent s to $E_t \cap G$ (at points where such a tangent exists) is equal to $|g_f| \cos(g_f, s)$, where g_f denotes the gradient of the function f .

If the real and imaginary parts of the mapping $T(z) = u(z) + iv(z)$ of the domain G are continuously differentiable, then, using Theorems 1 and 2, one obtains the equality:

$$\int_{t_1}^{t_2} l[T(E_t \cap G)] dt = \iint_{G \setminus F} \{|g_u|^2 \cos^2 |g_u, s| + |g_v|^2 \cos^2(g_v, s)\}^{1/2} dx dy, \quad (3)$$

where l is the sum of the lengths of the images of the arcs comprising the intersection $E_t \cap G$, and g_u and g_v are the gradients of the functions u and v , respectively.

From equality (3) follows the relation

$$\int_{t_1}^{t_2} l[T(E_t \cap G)] dt \leq \iint_{G \setminus F} (|u_x| + |u_y| + |v_x| + |v_y|) dx dy, \quad (4)$$

which, by means of a suitable smoothing process (see, for example, ⁽⁴⁾, p. 218), extends to continuous mappings with generalized first partial derivatives in G (⁽⁴⁾, p. 338). The class of such mappings is denoted by $D(G)$.

Inequality (4) shows that if the first generalized derivatives of the mapping $T(z) \in D(G)$ are summable on $G \setminus F$, then the quantity $l[T(E_t \cap G)]$ is finite for almost all $t \in (t_1, t_2)$. A consequence of (3) is the inequality

$$\int_{t_1}^{t_2} l[T(E_t \cap G)] dt \leq \left\{ m(G \setminus F) \cdot \iint_{G \setminus F} (|g_u|^2 + |g_v|^2) dx dy \right\}^{1/2}$$

(cf. ⁽⁵⁾, p. 178 or ⁽⁶⁾), which also remains valid for mappings $T \in D(G)$ and is nontrivial if the factors on the right are both finite.

Theorem 3. Let the Dirichlet integral

$$I_G(T) = \iint_G (|T_x|^2 + |T_y|^2) dx dy$$

of the mapping $T(z) \in D(G)$ be finite, i.e., let T belong to the class $BL(G)$ (see ⁽⁷⁾). Then

$$\int_{t_1}^{t_2} \frac{l^2[T(E_t \cap G)]}{l(E_t \cap G)} dt \leq \iint_{G \setminus F} (u_x^2 + v_x^2 + u_y^2 + v_y^2) dx dy, \quad (5)$$

where the integrand in the left-hand side should be regarded as equal to zero if the length $l(E_t \cap G) = \infty$ or $l[T(E_t \cap G)] = 0$.

It is easy to show, however, that relation (5) also has meaning for mappings $T(z) \in D(G)$ whose derivatives are summable on the set $G \setminus F$.

Let us note that if the set F consists of a single point $a \neq \infty$, then from (5) one immediately obtains the well-known inequality

$$\int_{t_1}^{t_2} l^2[T(E_t \cap G)] \frac{dt}{t} \leq 2\pi I_G(T),$$

which was used as the basis for the investigations in ^(5,7).

Next, let

$$T(w) = x(u, v) + iy(u, v)$$

be an (interior) quasiconformal (⁽⁸⁾, p. 24) mapping of the domain (or open set) Δ

in the z -plane, and $n(z)$ is the number of roots of the equation $h(w) = z$ lying in Δ . It can be shown that the function $n(z)$ is measurable. Put $p(t) = \int_{E_t} n(z) ds$, where ds is the element of arc length and $z \in E_t$.

The following proposition is a generalization of the well-known principle of length and area (see ^(9, 10)).

Theorem 4. Suppose that the function $T(w)$ carries out an (interior) Q -quasiconformal mapping of the open set Δ with area S_Δ , and $G = T(\Delta)$. Then

$$\int_{t_1}^{t_2} l^2[T^{-1}(E_t \cap G)] \frac{dt}{p(t)} \leq QS_\Delta, \quad (6)$$

where $l[T^{-1}(E_t \cap G)]$ is the total length of the curves in Δ constituting the preimage of the set $E_t \cap G$ (the expression under the integral on the left is regarded as equal to zero if $p(t) = \infty$ or $l[T^{-1}(E_t \cap G)] = 0$).

Using Theorem 3, the following boundary property is proved.

Theorem 5. Let $T(z) \in BL(G)$ be a topological mapping of the domain G (with boundary ∂G) onto a domain $\Delta = T(G)$ with rectifiable boundary. If $F \subset \partial G$ is a closed set and

$$\lim_{t \rightarrow 0} \frac{l(E_t \cap G)}{t} < \infty,$$

then the cluster set* $C_F(T)$ has length zero.

It would be interesting to compare this result with Theorems 3 and 4 of the paper ⁽¹¹⁾, which are close in character to the last theorem.

Tomsk State University
named after V. V. Kuibyshev

Received
17 XII 1965

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* The cluster set $C_N(T)$ is the totality of all limit points of all possible sequences of the form $\{T(z_n)\}$, where $z_n \in G$, $\lim_n z_n = z_0 \in N \subset \overline{G}$.

Note: Figure translations are in progress. See original paper for figures.

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