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Abstract

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MATHEMATICS

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ON THE THEORY OF CANONICAL SYSTEMS OF DIFFERENTIAL EQUATIONS OF POSITIVE TYPE WITH PERIODIC COEFFICIENTS

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In this note we use the notation and results of the works ⁽¹⁻³⁾. All terminology has been borrowed by us from ⁽³⁾.

We consider a real linear differential system with periodic coefficients of the form

$$dx/dt = JH(t)x \quad (H(t+T) = H(t)), \quad (1)$$

where x is an n -dimensional ($n = 2m$) vector function; $H(t)$ is a symmetric matrix function with elements from $L_1(0, T)$; $J = \text{diag}\{J_1, \dots, J_1\}$, where

$$J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $X(t)$ be the matriciant of equation (1), i.e., an absolutely continuous matrix function satisfying equation (1) and the initial condition $X(0) = I$. We note that this matrix is symplectic for all $t \in [0, T]$. It can be represented in the form

$$X(t) = e^{JQ(t)} \quad (Q = Q^*).$$

From the absolute continuity of $X(t)$ follows the absolute continuity of $Q(t)$.

Equation (1) is called an **equation of positive type** if $H(t) \geq 0$ ($0 \leq t \leq T$) and

$$\int_0^T H(t) dt > 0.$$

Following R. Bott ⁽⁵⁾, we shall call the matriciant of an equation of positive type a **positively directed symplectic curve**.

Main theorem. *Every stable equation of the form (1) of positive type can be continuously deformed into a split system of equations of the form*

$$dx_k/dt = J_1 H_k(t) x_k(t) \quad (k = 1, 2, \dots, m) \quad (2)$$

$(x_1, \dots, x_m)$ are 2-dimensional vectors)

with preservation (in the process of deformation) of stability and positivity.

The proof of this theorem is based on the representation of the Hamiltonian H of equation (1) through the eigenvalues and eigenvectors of the matrix $-iJQ(t)$.

If, for some $t = t_0$, the matrix $-iJQ(t)$ has a simple spectrum, then we shall call t_0 a **simple point**, and otherwise a **multiple point**.

In a neighborhood of a simple point the matrix $-iJQ$ has p pairs $(\omega, -\omega + 2k\pi)$ of real eigenvalues, q pairs $(\omega, -\omega = \bar{\omega})$ of numbers of imag-

* By a stable equation we mean an equation with a strongly stable monodromy matrix $X(T)$.

real eigenvalues and r quadruples $(\omega, \bar{\omega}, -\bar{\omega} + 2k\pi, -\omega + 2k\pi)$ of complex eigenvalues ($2p + 2q + 4r = n$). The corresponding n eigenvectors can be normalized and numbered so that the relation holds:

$$-iS^*JS = \hat{J}.$$

Here S is the matrix whose columns are the eigenvectors of the matrix $-iJQ(t)$, and

$$\hat{J} = \text{diag} \left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \dots}_{p \text{ blocks}}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots}_{q \text{ blocks}}, \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \dots}_{r \text{ pairs of blocks}} \right\}.$$

Theorem 1. In a neighborhood of a nonsingular point t_0 , the matrix H is representable in the form*

$$H = S^* \left(\frac{d\Omega}{dt} \hat{J} + E \circ K \right) S, \quad (3)$$

where $\Omega = \text{diag}\{\omega_1, \dots, \omega_n\}$ is the diagonal form of the matrix $-iJQ(t)$,

$$E = \left\| i(e^{i(\bar{\omega}_j - \omega_k)} - 1) \right\|_{k,j=1}^n, \quad K = -iS^*JdS/dt$$

is a skew-Hermitian matrix^{**}. Moreover,

$$e^{i\bar{\Omega}} = -Pe^{i\Omega}P, \quad \bar{S} = SP, \quad \bar{K} = -PKP,$$

where P is a permutation matrix:

$$P = \text{diag} \left\{ \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \dots, I_{2q}}_{p \text{ blocks}}; \underbrace{\begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}, \dots}_{q \text{ blocks}} \right\}$$

(I_s is the identity matrix of order s).

Conversely, suppose that in a neighborhood of the point t_0 the following are given:

1°. A collection of functions $\omega = \omega_k(t)$ ($k = 1, 2, \dots, n$), pairwise distinct for every t , and such that the matrix $\Omega = \text{diag}\{\omega_1, \dots, \omega_n\}$ has the property

$$e^{i\bar{\Omega}} = -Pe^{i\Omega}P.$$

2°. An arbitrary skew-Hermitian matrix $K(t)$, for which

$$\bar{K} = -PKP.$$

3°. A constant matrix S_0 , for which

$$-iS_0^*JS_0 = \hat{J}$$

and

$$\bar{S}_0 = S_0P.$$

Let $S(t)$ be the solution of the equation

$$dS/dt = S\hat{J}K,$$

satisfying the initial condition

$$S(t_0) = S_0.$$

If the Hamiltonian H of equation (1) has, in a neighborhood of the point t_0 , the form (3), then for the matrizant $X(t)$ in the same neighborhood the matrix $e^{i\Omega}$ is a diagonal form, and the matrix S is the matrix of eigenvectors.

The proof is carried out by direct calculation of the matrix

$$H = -J\frac{dX}{dt}X^{-1}$$

using the formula of Yu. L. Daletskii–S. G. Krein (4) for finding

$$\frac{dX}{dt} = \frac{d}{dt} e^{JQ(t)}.$$

From this, as a consequence, we immediately obtain a spectral characterization of a positively directed symplectic curve.

In order that a symplectic curve $X(t) = e^{JQ(t)}$ with pairwise distinct eigenvalues be positively directed, it is necessary and sufficient that the matrix

$$\mathfrak{H} = \frac{d\Omega}{dt} \hat{J} + E \circ K$$

be ≥ 0 for all t , and on some set of positive measure $\mathfrak{H} > 0$.

This criterion and representation (3) make it possible to construct a deformation of system (1) into the split system (2).

Without loss of generality one may assume that on the interval $[0, T]$ there is only a finite set of multiple points

$$t_1 < t_2 < \dots < t_N.$$

The points t_k ($k = 1, 2, \dots, N$) divide $[0, T]$ into a number of subintervals without multiple points.

* Here $E \circ K$ is the Hadamard (elementwise) product of E and K ,

** The last relation is equivalent to the relation $dS/dt = S\hat{J}K$.

For convenience of notation, we shall assume that on some fixed interval the eigenvalues ω_k ($k = 1, 2, \dots, n$) are numbered as follows: first the p real pairs are numbered, then the q purely imaginary pairs, and then the r complex quadruples. Write H in the form (3) and set

$$R = \text{diag}\{\underbrace{1, \dots, 1}_{2p}, \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{q+2r \text{ blocks}}\},$$

$$K_1 = K \circ R, \quad K_2 = K - K_1 \quad (K_1^* = -K_1, \quad K_2^* = -K_2).$$

Rewrite the matrix H in the form $H = S^*(\mathfrak{H}_1 + E \circ K_2)S$, where

$$\mathfrak{H}_1 = \frac{d\Omega}{dt} \hat{J} + E \circ K_1 \geq 0.$$

The deformation of system (1) into (2) on each selected interval is carried out in two steps.

I. We deform the eigenvectors of the matrix $-iJQ$ (without touching its eigenvalues) so that $\mathfrak{H}$ is simplified. Namely, put $K_\alpha = K_1 + (1 - \alpha)K_2$ ($0 \leq \alpha \leq 1$). Then $\mathfrak{H}_\alpha = \mathfrak{H}_1 + (1 - \alpha)E \circ K_2 = (1 - \alpha)\mathfrak{H} + \alpha\mathfrak{H}_1 \geq 0$. For $\alpha = 1$ we arrive at the block-diagonal matrix* $\mathfrak{H}_1$: to a real eigenvalue ω there corresponds the diagonal element ω' , and to a complex-conjugate pair $(\omega, \bar{\omega})$ there corresponds a diagonal block of the form

$$\begin{pmatrix} i(e^{2\text{Im}\omega} - 1)k_+ & \omega' \\ \bar{\omega}' & i(e^{-2\text{Im}\omega} - 1)k_- \end{pmatrix}.$$

II. We deform the eigenvalues of the matrix $-iJQ$ so that the complex eigenvalues $\pm\omega$, $\pm\bar{\omega}$ pass onto the real axis. At the same time we deform the eigenvectors so that $\mathfrak{H}$ remains nonnegative in the process of deformation. Namely, if ω_0 leaves the real axis at $t = t_k$ and returns to it at $t = t_l$, put

$$\omega_0(t; \beta) = (1 - \beta)\omega_0(t) + \beta\omega_0(t_l), \quad k_\pm(t; \beta) = \frac{(1 - \beta)(e^{\pm 2\text{Im}\omega_0} - 1)}{e^{\pm 2\text{Im}(1 - \beta)\omega_0} - 1} k_\pm$$

$$(0 \leq \beta \leq 1; t_k \leq t \leq t_l).$$

For $\beta = 1$ system (1) on the indicated interval passes into (2). For $t > t_l$ the real increasing differentiable function $\omega_0(t)$ may be deformed arbitrarily, preserving reality, increase, and differentiability, taking care only that for all $\beta \in [0, 1]$ continuity be satisfied at the junction: $\omega_0(t_l - 0) = \omega_0(t_l + 0)$. Repeating this process, we arrive at (2). A detailed study of the features arising at multiple points shows that in the process of deformation $H(t)$ remains summable.

Corollary. *Two positively directed symplectic curves from one stability region can be deformed one into the other with preservation of positive directedness, without leaving the indicated stability region.*

Consider the spectrum of the monodromy matrix $X(T)$. Moving along the upper semicircle of the unit circle from 1 to -1 , we shall write 0 when we encounter a multiplier of the first kind, and 1 when we encounter a multiplier of the second kind (each of the numbers 0, 1 is written as many times as is the multiplicity of the corresponding multiplier). As a result we obtain an n -digit binary number —the **multiplier type** of the corresponding system (1).

* It does not follow from this that system (1) has already been reduced to the form (2). It can be shown that the matrix S in this case has not yet split into second-order blocks.

Theorem 2. The index of the stability domain containing system (1) is equal to the number of pairs of characteristic numbers of the antiperiodic boundary-value problem

$$dx/dt = \lambda JHx, \quad x(T) = -x(0), \quad (4)$$

lying between $\lambda = 0$ and $\lambda = 1$, minus the number of ones in the binary notation of the multiplier type of this equation.

For $n = 2$, the assertion of the theorem is easily obtained from the considerations of (1). In the case of arbitrary n , it remains to note that under the deformation described above the multiplier type, the index, and the number of characteristic numbers of the boundary-value problem (4) between $\lambda = 0$ and $\lambda = 1$ do not change.

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REFERENCES

- ¹ M. G. Krein, in: *In Memory of A. A. Andronov*, Moscow, 1955, p. 443.
- ² I. M. Gelfand, V. B. Lidskii, *Uspekhi Mat. Nauk*, 10, 1 (63), 3 (1955).
- ³ M. G. Krein, V. A. Yakubovich, *Proceedings of the International Symposium on Nonlinear Oscillations*, 1, Kiev, 1963, p. 277.
- ⁴ Yu. L. Daletskii, *Izv. Vyssh. Uchebn. Zaved., Matematika*, No. 2, 52 (1962).
- ⁵ R. Bott, *Comm. Pure and Appl. Math.*, 70, 313 (1959).

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