

ON SPATIAL GROUPS OF SIMILARITY SYMMETRY

A. M. Zamorzaev

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.09544>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.45

CRYSTALLOGRAPHY

A. M. ZAMORZAEV

ON SPATIAL GROUPS OF SIMILARITY SYMMETRY

(Presented by Academician A. V. Shubnikov on 4 VI 1965)

1. In 1960 A. V. Shubnikov introduced into the Soviet literature the concept of groups of similarity symmetry ⁽¹⁾. A refinement of this concept, a more complete study of two-dimensional groups of similarity symmetry, and the transfer to them of the ideas of antisymmetry ⁽²⁾ and of some of its generalizations ⁽³⁾ were carried out in ⁽⁴⁾.

A similarity transformation P with similarity coefficient k in an n -dimensional Euclidean space is defined by the property $A'B' = k \cdot AB$, where $A' = P(A)$, $B' = P(B)$, and the points A and B are arbitrary. If $k \neq 1$, then P has a unique fixed point O and is the product of a homothety K with center O and coefficient k^* by a motion s (of the first or second kind) preserving the fixed point O ; moreover, K and s commute. These facts were proved in ⁽⁵⁾ for $n \leq 3$ and are proved analogously for any n .

The point O , fixed under a given similarity transformation P with $k \neq 1$, will be called its singular point ⁽⁴⁾. To it converges any sequence of points $A_1, A_2, \dots, A_n, \dots$, where, for $i = 1, 2, \dots$, either $A_{i+1} = P(A_i)$ (if $k < 1$) or $A_{i+1} = P^{-1}(A_i)$. All nonidentity transformations of the cyclic group $\{P\}$ have O as their singular point.

We shall call an n -dimensional group of similarity symmetry a group of similarity transformations in n -dimensional space having the following properties: a) it contains at least one transformation P with coefficient $k \neq 1$; b) at least one point of the space is isolated in the infinite class of its images under all transformations of the given group. The singular point of the transformation P will be called a singular point of the group of similarity symmetry.

The uniqueness of a singular point for any group of similarity symmetry is not included in the definition. It follows from the theorem expressing the basic property of a group of similarity symmetry.

Theorem. *Every transformation of an n -dimensional group of similarity symmetry leaves fixed any of its singular points.*

This theorem was proved in ⁽⁴⁾ for $n = 2$. There it was also shown that the derivation of all two-dimensional groups of similarity symmetry (and antisymmetry) can be reduced to the study of groups of symmetry and antisymmetry of rods, better known in geometrical crystallography ⁽⁶⁻⁸⁾.

The method of proof given for $n = 2$ ⁽⁴⁾ of the basic property of a group of similarity symmetry does not carry over directly to the case $n \geq 3$. In the present work this theorem is proved completely for the three-dimensional case. The method of proof proposed here already carries over completely to any n -dimensional case.

2. Let us set out some preliminary considerations for $n = 3$. As indicated in item 1, if P is any similarity transformation with coefficient $k \neq 1$, then $P = Ks$, where K is a homothety with center O and coefficient k (which we shall briefly denote by the symbol: $K \sim O, k$), and s is a motion, with $s(O) = O$. Corresponding to all types of motions s , the transformation P has four types:

* $M' = K(M)$ is defined by the property: $\overrightarrow{OM'} = k \cdot \overrightarrow{OM}$.

- 1) a homothety $K \sim O, k$;
- 2) a screw motion ^(1,4) $L = Kv$, where v is a rotation through an angle φ about the axis* with directing unit vector $\mathbf{l}$ (φ is the magnitude of the oriented angle, whose orientation is coordinated with the direction of the vector $\mathbf{l}$); in brief notation: $v \sim \mathbf{l}, \varphi$ (at the same time $v \sim -\mathbf{l}, -\varphi$); $L \sim O, k, \mathbf{l}, \varphi$ ($L \sim O, k, -\mathbf{l}, -\varphi$);
- 3) a homothetic reflection ⁽⁴⁾ $M = Km$, where m is reflection in a plane*; $M^2 = K^2 = K_1 \sim O, k^2$;
- 4) a screw reflection ⁽⁴⁾ $\bar{L} = K\bar{v}$, where $\bar{v} = vm = mv$ is a rotatory reflection; $\bar{L} = Lm, \bar{L}^2 = L^2 = L_1 \sim O, k^2, \mathbf{l}, 2\varphi$.

For the proof of the theorem we shall need two lemmas on screw motions.

Lemma 1. *Let $L \sim O, k, \mathbf{l}, \varphi$ and $L_1 \sim O, k, \mathbf{l}_1, \varphi$; then for any point M and its $(L_1^{-1}L)$ -image M_1 the formula holds*

$$MM_1 \leq 2 \cdot OM \cdot \widehat{(\mathbf{l}, \mathbf{l}_1)}. \quad (1)$$

Indeed, decompose each of the screw rotations into a homothety and a rotation:

$$L = Kv, \quad L_1 = Kv_1 \quad (K \sim O, \varphi; v \sim \mathbf{l}, \varphi; v_1 \sim \mathbf{l}_1, \varphi).$$

Then $L_1^{-1}L = v_1^{-1}v = v_2 \sim \mathbf{l}_2, \varphi_2$, with $\varphi_2 \leq 2\widehat{(\mathbf{l}, \mathbf{l}_1)}$ by the rule, known in the classical theory of symmetry, for multiplying rotations ^(2,6). By obvious geometric considerations $MM_1 \leq OM \cdot \varphi_2$, whence formula (1) follows.

Lemma 2. Let $L \sim O, k, \mathbf{l}, \varphi$ and $L_1 \sim O_1, k, \mathbf{l}, \varphi$; then for any point M and its (L_1^{-1}) -image M_1 the formula holds

$$MM_1 \leq (1 + 1/k) \cdot OO_1. \quad (2)$$

In fact, let the point $M^* = L(M)$; then $M_1 = L_1^{-1}(M^*)$; further, let t be translation by the vector $\overrightarrow{O_1O}$; then $L_1^{-1} = t^{-1}L^{-1}t$; let $N^* = t(M^*)$, then

$$M^*N^* = OO_1, \quad (2_1)$$

and, finally, let $N = L^{-1}(N^*)$, then

$$MN = \frac{1}{k} \cdot M^*N^*, \quad (2_2)$$

while $M_1 = t^{-1}L^{-1}t(M^*) = t^{-1}L^{-1}(N^*) = t^{-1}(N)$, i.e. $N = t(M_1)$; hence

$$NM_1 = OO_1. \quad (2_3)$$

Formula (2) follows from the triangle inequality and formulas (2₁)–(2₃).

3. We pass to the proof of the theorem. Let a three-dimensional group of similarity symmetry be given, and let P be an arbitrary transformation of it with coefficient $k \neq 1$ (without loss of generality, we take $k < 1$), O its fixed point, and Q any other transformation from the given group. We assert that $Q(O) = O$.

We shall prove this by contradiction: the assumption that the point $O_1 = Q(O)$ does not coincide with O contradicts discreteness of the similarity symmetry group—the requirement b) in its definition.

Main case. $P = L \sim O, k, \mathbf{l}, \varphi$ (a screw motion). Let the point $O_1 = Q(O)$ be distinct from O ; then $P_1 = QPQ^{-1}$ is a screw motion $L_1 \sim O_1, k, \mathbf{l}_1, \varphi$, where the vector $\mathbf{l}_1 = \pm Q(\mathbf{l})$ **.

* Passing through the point O .

** The minus sign is put if Q is a transformation of the 2nd kind, since in this case the orientation of the angle φ in the screw motion L_1 is not coordinated with the direction of the vector $Q(\mathbf{l})$.

Denote: $L_2 = LL_1L^{-1}, \dots, L_{n+1} = LL_nL^{-1}, \dots$; for $i = 2, 3, \dots$ each $L_i \sim O_i, k, \mathbf{l}_i, \varphi$, where $O_i = L(O_{i-1})$, $\mathbf{l}_i = L(\mathbf{l}_{i-1})$. By the choice of the point O , the distance $OO_n \rightarrow 0$ as $n \rightarrow \infty$.

The sequence of vectors $\mathbf{l}_1, \mathbf{l}_2, \dots, \mathbf{l}_n, \dots$ is not convergent, but it is bounded (all the vectors $\mathbf{l}_i$ are unit vectors), and, by the Bolzano-Weierstrass principle, one can extract from it a convergent subsequence $\mathbf{l}_{i_1}, \mathbf{l}_{i_2}, \dots, \mathbf{l}_{i_n}, \dots$ (in general, not converging to the vector $\mathbf{l}$).

We now choose an arbitrary point M . We assert that for every $\varepsilon > 0$ there exists a natural number n such that for any natural m , from the condition

$$M' = L_{i_{n+m}}^{-1} L_{i_n}(M) \quad (3)$$

it follows that

$$MM' < \varepsilon, \quad (4)$$

which will lead to a contradiction with b).

Let $\varepsilon > 0$ be given. Choose $\varepsilon_1 > 0, \varepsilon_2 > 0$ so that

$$\varepsilon = 2[R\varepsilon_1 + (1 + 1/k)\varepsilon_2(1 + \varepsilon_1)], \quad (4_1)$$

where $R = OM$; then choose n_1, n_2 so that for all $n > n_1, m \geq 1$ the requirement

$$(\mathbf{l}_{i_n}, \widehat{\mathbf{l}_{i_{n+m}}}) < \varepsilon_1, \quad (4_2)$$

is satisfied, and for all $n > n_2, m \geq 1$ the requirement

$$OO_{i_n} < \varepsilon_2, \quad OO_{i_{n+m}} < \varepsilon_2 \quad (4_3)$$

is satisfied.

Choose an arbitrary $n > \max(n_1, n_2)$; we shall prove that it is the desired one. For this purpose introduce auxiliary screw motions L^* and L^{**} (they need not belong to the given group): $L^* \sim O, k, \mathbf{l}_{i_n}, \varphi$; $L^{**} \sim O, k, \mathbf{l}_{i_{n+m}}, \varphi$.

Now let $M^* = (L^*)^{-1}L_{i_n}(M)$, and $M^{**} = (L^{**})^{-1}L^*(M^*)$, if M' is chosen according to (3), then $M' = L_{i_{n+m}}^{-1}L^{**}(M^{**})$. Then, by Lemma 1 and condition (4_2), $M^*M^{**} < 2 \cdot OM^* \cdot \varepsilon_1$; by Lemma 2 and condition (4_3):

$$MM^* < (1 + 1/k)\varepsilon_2, \quad M^{**}M' < (1 + 1/k)\varepsilon_2; \quad (4_4)$$

hence $OM^* < R + (1 + 1/k)\varepsilon_2$, and therefore

$$M^*M^{**} < 2[R + (1 + 1/k)\varepsilon_2]\varepsilon_1. \quad (4_5)$$

Applying the triangle inequality and equality (4_1), from conditions (4_4) and (4_5) we obtain (4), i.e. the chosen n is indeed the desired one.

Since the point M was chosen arbitrarily, requirement b) is violated. For the case $P = L$ the theorem is proved.

The remaining three cases present no difficulty. The case $P = K$ may be regarded as particular with respect to the basic one ($\varphi = 0$, $\mathbf{l}$ is chosen arbitrarily), while in fact the reasoning can be considerably simplified. The case $P = M$ reduces to the preceding one, and $P = \bar{L}$ to the basic one by replacing P by P^2 . The theorem is completely proved.

4. As is not hard to see, the basic idea of the proof is suitable for any n (but for $n = 2$ there is no need to extract a subsequence). In a space of dimension $n > 3$ the matrix of any similarity transformation with coefficient $k \neq 1$ can (by choosing an orthonormal frame) be brought to the form $k \cdot A$, where the matrix A either has diagonal form with the numbers ± 1 on the main diagonal, or block-diagonal form, where m blocks $A_1, \dots, A_m$ are matrices of the form

$$A_i = \begin{pmatrix} \cos \varphi_i & -\sin \varphi_i \\ \sin \varphi_i & \cos \varphi_i \end{pmatrix} \quad (i = 1, \dots, m),$$

and the remaining $n - 2m$ diagonal elements are likewise equal to ± 1 . Replacing P by P^2 , one can eliminate the minus signs before the units. The basic form of P for the proof will be the "screw motion" $L \sim O, k, (l_1, \dots, l_m), (\varphi_1, \dots, \varphi_m)$, where $l_1, \dots, l_m$ are bivectors characterizing the "axis" of a compound rotation through the angles $\varphi_1, \dots, \varphi_m$ about a system of m distinct $(n - 2)$ -dimensional planes; all considerations connected with removing singularities of a point, constructing the sequence $L_1, L_2, \dots$, and applying the Bolzano-Weierstrass principle remain valid.

It is not fundamentally difficult to construct, as was done for $n = 2$ ⁽⁴⁾, such a homeomorphic mapping of the hypersurface of the "straight spherical cylinder" in $(n + 1)$ -dimensional space* onto a hyperplane with one point removed; this naturally entails an isomorphic mapping of the symmetry group of the "directed" cylindrical hypersurface (i.e., with mutually nonequivalent opposite directions along the generators) onto the n -dimensional group of similarity transformations preserving the "puncture" of the hyperplane. It is enough to generalize directly the considerations of p. 3 of ⁽⁴⁾. Consequently, it is expedient to investigate in parallel the n -dimensional symmetry groups of similarity and the $(n + 1)$ -dimensional linear ⁽⁸⁾ groups of ordinary symmetry, and to transfer to both the idea of antisymmetry, etc.

Kishinev State
University

Received
4 VI 1965

CITED LITERATURE

- ¹ A. V. Shubnikov, *Kristallografiya*, 5, No. 4, 489 (1960).
- ² A. V. Shubnikov, *Symmetry and Antisymmetry of Finite Figures*, Moscow, 1951.
- ³ A. M. Zamorzaev, E. I. Sokolov, *Kristallografiya*, 2, No. 1, 9 (1957).
- ⁴ E. I. Talyarskii, A. M. Zamorzaev, *Kristallografiya*, 8, No. 5, 691 (1963).
- ⁵ P. S. Modenov, A. S. Parkhomenko, *Geometric Transformations*, Moscow, 1951.
- ⁶ A. V. Shubnikov, *Symmetry*, Moscow, 1940.
- ⁷ A. V. Shubnikov, *Atlas of Crystallographic Symmetry Groups*, Moscow-Leningrad, 1946.
- ⁸ N. N. Neronova, N. V. Belov, *Kristallografiya*, 6, No. 1, 3 (1961).

* The generator of the cylinder is a straight line, and the directrix is an $(n - 1)$ -dimensional surface of a sphere in a hyperplane perpendicular to the generator.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.