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S. Z. Rafal'son

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Abstract

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MATHEMATICS

S. Z. RAFALSON

ON AN ASYMPTOTIC FORMULA IN THE THEORY OF ORTHOGONAL POLYNOMIALS

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1. It is known that the derivatives of the orthogonal Jacobi polynomials are again orthogonal Jacobi polynomials, but with different indices. Namely, the equality holds

$$dJ_n^{(\alpha, \beta)}(x)/dx = \sqrt{n(n + \alpha + \beta + 1)} J_{n-1}^{(\alpha+1, \beta+1)}(x), \quad n \geq 1 \quad (1)$$

(here $J_m^{(\alpha, \beta)}(x)$ is the normalized Jacobi polynomial of degree m with indices α and β).

It is of interest to determine how equality (1) changes if, instead of the weight $(1-x)^\alpha(1+x)^\beta$, one considers the more general weight $q(x)(1-x)^\alpha(1+x)^\beta$, where the function $q(x)$ satisfies certain conditions. We shall show that in this case equality (1) is replaced by an asymptotic equality.

Theorem 1. Let $\{\omega_n(x)\}_0^\infty$ be a system of polynomials orthonormal with weight $p(x) = (1-x)^\alpha(1+x)^\beta q(x)$, where $\alpha, \beta \geq -1/2$, and let the function $q(x)$ satisfy the following conditions: 1) $q(x) \geq m > 0$, $x \in [-1, 1]$; 2) $q(x) \in \text{Lip } 1$. Let $\{\varphi_n(x)\}_0^\infty$ be a system of polynomials orthonormal with weight $(1-x^2)p(x)$.

The asymptotic formula is valid

$$\omega'_n(x) = a_n \varphi_{n-1}(x) + O(\ln n), \quad (2)$$

where $a_n \asymp n$, and the constant occurring in $O(\ln n)$ does not depend on $x \in [-1+h, 1-h]$, $0 < h < 1$.

On the entire interval $[-1, 1]$ the asymptotic equality holds

$$\omega'_n(x) = a_n \varphi_{n-1}(x) + O(n^{\sigma+3/2} \ln n), \quad (3)$$

where $\sigma = \max\{\alpha, \beta\}$; the constant occurring in $O(n^{\sigma+3/2} \ln n)$ does not depend on $x \in [-1, 1]$.

Proof. We first prove equality (2). Represent $\omega'_n(x)$ in the form of a linear combination of the polynomials $\{\varphi_k(x)\}_0^{n-1}$:

$$\omega'_n(x) = a_n \varphi_{n-1}(x) + \sum_{k=0}^{n-2} c_k \varphi_k(x). \quad (4)$$

It is easy to obtain the following expression for c_k :

$$c_k = - \int_{-1}^1 (1-t)^{\alpha+1} (1+t)^{\beta+1} q'(t) \omega_n(t) \varphi_k(t) dt, \quad 0 \leq k \leq n-2. \quad (5)$$

The sum $\sum_{k=0}^{n-2} c_k \varphi_k(x)$, with the aid of the expressions obtained for c_k , is represented in the form

$$I = \sum_{k=0}^{n-2} c_k \varphi_k(x) = - \int_{-1}^1 (1-t)^{\alpha+1} (1+t)^{\beta+1} q'(t) \omega_n(t) \sum_{k=0}^{n-2} \varphi_k(t) \varphi_k(x) dt. \quad (6)$$

Applying the Christoffel-Darboux formula and taking into account that almost everywhere $q'(t) = O(1)$, we obtain:

$$|I| = O(1) \int_{-1}^1 (1-t)^{\alpha+1} (1+t)^{\beta+1} |\omega_n(t)| \left| \frac{\varphi_{n-2}(x) \varphi_{n-1}(t) - \varphi_{n-2}(t) \varphi_{n-1}(x)}{t-x} \right| dt. \quad (7)$$

We split the integral on the right-hand side of equality (7) into three integrals according to the scheme

$$\int_{-1}^1 = \int_{-1}^{x-1/n} + \int_{x-1/n}^{x+1/n} + \int_{x+1/n}^1 = I_1 + I_2 + I_3.$$

Let us first estimate the integrals I_1 and I_3 . Applying Korov's theorem (see (3), p. 169), we obtain:

$$\begin{aligned} I_1 = & \int_{-1}^{x-1/n} (1-t)^{\alpha+1} (1+t)^{\beta+1} [O(1)|J_n^{(\alpha,\beta)}(t)| + O(1)|J_{n-1}^{(\alpha,\beta)}(t)|] |t-x|^{-1} \\ & \times \left\{ |\varphi_{n-2}(x)| [O(1)|J_{n-1}^{(\alpha+1,\beta+1)}(t)| + O(1)|J_{n-2}^{(\alpha+1,\beta+1)}(t)|] \right. \\ & \left. + |\varphi_{n-1}(x)| [O(1)|J_{n-2}^{(\alpha+1,\beta+1)}(t)| + O(1)|J_{n-3}^{(\alpha+1,\beta+1)}(t)|] \right\} dt. \end{aligned} \quad (8)$$

It is enough to estimate one of the integrals entering the right-hand side of (8), for example the first of them, since the remaining integrals are estimated analogously. For this, note, first, that for $-1 + h \leq x \leq 1 - h$, $0 < h < 1$, the equality $J_m^{(\alpha+1, \beta+1)}(x) = O(1)$ holds (see ⁽³⁾, pp. 80 and 204), whence

$$\varphi_m(x) = O(1), \quad -1 + h \leq x \leq 1 - h, \quad 0 < h < 1, \quad (9)$$

and, secondly, we use the relations

$$\begin{aligned} (1-t)^{\alpha/2+1/4}(1+t)^{\beta/2+1/4}|J_m^{(\alpha, \beta)}(t)| &= O(1), \\ (1-t)^{\alpha/2+3/4}(1+t)^{\beta/2+3/4}|J_m^{(\alpha+1, \beta+1)}(t)| &= O(1), \end{aligned} \quad (10)$$

which are valid since $\alpha, \beta \geq -1/2$ (see ⁽²⁾, pp. 57, 69, 70; see also ⁽³⁾, p. 177). Then we obtain

$$\begin{aligned} \int_{-1}^{x^{-1/n}} (1-t)^{\alpha+1}(1+t)^{\beta+1} \frac{|J_n^{(\alpha, \beta)}(t)| |\varphi_{n-2}(x)| |J_{n-1}^{(\alpha+1, \beta+1)}(t)|}{|t-x|} dt = \\ = O(1) \int_{-1}^{x^{-1/n}} \frac{dt}{|t-x|} = O(\ln n). \end{aligned} \quad (11)$$

Thus, $I_1 = O(\ln n)$. It is proved analogously that $I_3 = O(\ln n)$.

Consider I_2 . With the aid of Korov's theorem and equalities (9) and (10), it is not difficult to obtain that $I_2 = O(1)$. Thus, $I = O(\ln n)$.

Denoting by $\beta_{n,0}$ the leading coefficient of the polynomial $\omega_n(x)$ ($\beta_{n,0} > 0$), and by $\gamma_{n-1,0}$ the leading coefficient of the polynomial $\varphi_{n-1}(x)$ ($\gamma_{n-1,0} > 0$), from equality (4) we have

$$a_n = n\beta_{n,0}/\gamma_{n-1,0}. \quad (12)$$

Under the conditions of the theorem the following equalities hold

$$\beta_{n,0} \simeq \pi^{-1/2} \cdot 2^n \exp \left\{ -\frac{1}{2\pi} \int_{-1}^1 \ln p(x) \frac{dx}{\sqrt{1-x^2}} \right\}, \quad (13)$$

$$\gamma_{n-1,0} \simeq \pi^{-1/2} \cdot 2^{n-1} \exp \left\{ -\frac{1}{2\pi} \int_{-1}^1 \ln[p(x)(1-x^2)] \frac{dx}{\sqrt{1-x^2}} \right\} \quad (14)$$

(see ⁽³⁾, p. 317).

From equalities (12), (13), and (14) it follows easily that $\alpha_n \simeq n$. Equality (2) is proved. Equality (3) is proved similarly; one need only take into account the equality $\|J_n^{(\alpha+1, \beta+1)}(x)\|_{C([-1, 1])} = O(n^{\sigma+3/2})$.

Corollary. The following equalities hold

$$\omega_n^{(k)}(x) = \gamma_n^{(k)} \delta_{n-k}(x) + O(n^{k-1} \ln n) \quad (15)$$

(here $-1+h \leq x \leq 1-h$, $0 < h < 1$; the constant entering into $O(n^{k-1} \ln n)$ does not depend on $x \in [-1+h, 1-h]$; $\gamma_n^{(k)} \simeq n^k$; $\delta_{n-k}(x)$ is a polynomial of degree $n-k$ from the sequence of polynomials orthonormal with weight $p(x)(1-x^2)^k$);

$$\omega_n^{(k)}(x) = \gamma_n^{(k)} \delta_{n-k}(x) + O(n^{\sigma+3/2+2(k-1)} \ln n) \quad (16)$$

(here $-1 \leq x \leq 1$; the constant entering into $O(n^{\sigma+3/2+2(k-1)} \ln n)$ does not depend on $x \in [-1, 1]$).

2. For a system of polynomials $\{\omega_k(x)\}$ satisfying the conditions of Theorem 1, the following holds.

Theorem 2. If

$$\sum_{k=1}^{\infty} c_k^2 k^{2j} \ln^2 k < \infty,$$

then the series

$$\sum_{k=1}^{\infty} c_k \omega_k^{(j)}(x)$$

converges almost everywhere on the interval $[-1, 1]$, $j = 1, 2, \dots$.

Corollary 1. If

$$\sum_{k=1}^{\infty} c_k^2 < \infty,$$

then almost everywhere on the interval $[-1, 1]$ the equality

$$s_n(x) = \sum_{k=1}^n c_k \omega_k'(x) = o_x(n \ln n)$$

holds.

Corollary 2. If $\lambda_n \uparrow \infty$, then from the condition

$$\sum_{n=1}^{\infty} \left[c_n^2 / \lambda_n \sum_{k=1}^n c_k^2 \right] < \infty$$

it follows that almost everywhere on the interval $[-1, 1]$

$$s_n(x) = o_x \left[\left(\lambda_n (\ln n)^2 n^2 \sum_{k=1}^n c_k^2 \right)^{1/2} \right].$$

3. **Definition.** A system of functions $\{A_n(x)\}_1^\infty$ will be called **quasi-orthonormal with weight $r(x)$ on the interval $[a, b]$** if, for any real numbers $c_1, c_2, \dots, c_n$, the equality

$$\int_a^b r(x) \left[\sum_{k=1}^n c_k A_k(x) \right]^2 dx = O(1) \sum_{k=1}^n c_k^2,$$

holds, where $O(1)$ depends neither on n nor on the numbers $c_1, c_2, \dots, c_n$.

It is clear that a system of functions orthonormal with weight $r(x)$ on $[a, b]$ is quasi-orthonormal. It is also easy to see that if $\{A_n(x)\}_1^\infty$ —

orthonormal with weight $r(x)$ on $[a, b]$, then the system of functions

$$\left\{ \sum_{k=0}^j \alpha_n^{(k)} A_{n-k}(x) \right\}_{n=j+1}^\infty,$$

where $\alpha_n^{(0)} = O(1)$, $\alpha_n^{(1)} = O(1)$, $\dots$, $\alpha_n^{(j)} = O(1)$, is a quasi-orthonormal system of functions with weight $r(x)$ on the interval $[a, b]$.

Theorem 3. Let $\{\omega_n(x)\}_0^\infty$ be a system of polynomials satisfying the conditions of Theorem 1, and let the constants α_k ($k = 1, 2, \dots$) be taken from equality (2).

The system of polynomials $\{\omega'_n(x)/\alpha_n\}_1^\infty$ is quasi-orthonormal with weight $p(x)(1-x^2)$ on the interval $[-1+h, 1-h]$, $0 < h < 1$.

For quasi-orthonormal systems the Menshov-Rademacher theorem, the theorem on the equivalence of A -summability and (C, γ) -summability, $\gamma > 0$, etc., are valid.

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Leningrad Financial-Economic Institute
named after N. A. Voznesensky

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Note: Figure translations are in progress. See original paper for figures.

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