

A CONTINUOUS MEDIUM WITH SPINOR AND VECTOR CHARACTERISTICS

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Abstract

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Mechanics

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A CONTINUOUS MEDIUM WITH SPINOR AND VECTOR CHARACTERISTICS

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1. Let there be given a four-dimensional Riemannian space G with metric tensor g_{ij} , specified in some holonomic coordinate system (x^i) , and in it a nonholonomic coordinate system $(\tilde{x}^i)$, consisting of frames with metric

$$ds^2 = \tilde{g}_{ij} d\tilde{x}^i d\tilde{x}^j = -d\tilde{x}^{12} - d\tilde{x}^{22} - d\tilde{x}^{32} + d\tilde{x}^{42},$$

where at each point of the space G the relations

$$d\tilde{x}^i = a_j^i dx^j, \quad dx^i = b_j^i d\tilde{x}^j, \quad (1)$$

hold, where $\|a_j^i\| = \|b_j^i\|^{-1}$, with sufficiently smooth functions b_j^i of x^i . From the equality

$$b_k^i b_l^j g_{ij} = \tilde{g}_{kl} \quad (2)$$

$\|b_k^i\|$ is determined up to a Lorentz transformation of the frame. A spinor field ψ_a ($a = 1, 2, 3, 4$), as a field of quantities transforming according to a representation of the Lorentz group of frame rotations, which we shall suppose to be given by constant Dirac matrices $\tilde{\gamma}_b^{i,a} * (1)$, satisfying the condition

$$\tilde{\gamma}_{i,a}^c \tilde{\gamma}_{j,c}^b + \tilde{\gamma}_{j,a}^c \tilde{\gamma}_{i,c}^b = 2\tilde{g}_{ij} \delta_b^a, \quad (3)$$

is defined in the nonholonomic coordinate system; therefore parallel transport of spinors is generated by parallel transport of frames (2).

Let us transport the frame from the point x^i to the point $x^i + dx^i$ in parallel, and let $\delta_j^i + d\tilde{\omega}_j^i$ ($d\tilde{\omega}^{ij} = -d\tilde{\omega}^{ji}$) be its infinitesimal Lorentz transformation; then the corresponding spinor transformation $\psi'_a = S_a^b \psi_b$ gives

$$\psi'_a - \psi_a = d\psi_a = \frac{1}{4} \tilde{\sigma}_{ij,a}^b \psi_b d\tilde{\omega}^{ij}, \quad (4)$$

where

$$\tilde{\sigma}_{ij,a}^b = \frac{1}{2} (\tilde{\gamma}_{i,b}^c \tilde{\gamma}_{j,c}^a - \tilde{\gamma}_{j,b}^c \tilde{\gamma}_{i,c}^a).$$

Since, according to parallel transport of a vector,

$$d\tilde{\omega}_j^i = \tilde{\Gamma}_{kj}^i d\tilde{x}^k = \left(a_l^i b_k^p b_j^q \Gamma_{pq}^l - b_k^p b_j^q \frac{\partial a_l^i}{\partial x^p} \right) a_m^k dx^m,$$

we obtain, for the increment of the spinor components under parallel transport, the formula

$$d\psi_a = \Gamma_{k,a}^b \psi_b dx^k, \quad \Gamma_{k,b}^a = \frac{1}{4} \tilde{\sigma}_{p,b}^{la} \left(b_s^p \frac{\partial a_l^s}{\partial x^k} - \Gamma_{kl}^p \right). \quad (5)$$

Consider the conjugate spinor ψ^{+a} , and let lowering and raising of indices be performed with the aid of the invariant fundamental spin-tensor e^{ab**} , contraction always being over the second index; then, since $\psi^{+a}\psi_a$ is a scalar, if there are no reflections along $\tilde{x}^4$ (1), we obtain the formula for parallel transport for ψ^{+a} :

$$d\psi^{+b} = -\psi^{+a} \Gamma_{k,a}^b dx^k. \quad (5')$$

* The letters a, b denote spinor indices; the sign $\sim$ indicates that the quantity is referred to the system $(\tilde{x}^i)$.

** e^{ab} satisfies the relation

$$e^{ab} \gamma_b^i e_{cd} = \gamma_{,d}^{i,a}.$$

Parallel transport of spinors makes it possible to introduce covariant differentiation of spintensors, denoted by the symbol ∇_i . As is known, $\gamma_{i,b}^d$ may be regarded as components of a vector with respect to the index i and as components of a spinor with respect to the indices a and b . It is easy to verify that

$$\nabla_p \gamma_{i,b}^a = \partial \gamma_{i,b}^a / \partial x^p - \Gamma_{pi}^s \gamma_{s,b}^a + \gamma_{i,b}^c \Gamma_{p,c}^a - \Gamma_{p,b}^c \gamma_{i,c}^a = 0. \quad (6)$$

Using (3), (5), and (6), one can derive that

$$\nabla_i \nabla_j \psi_a - \nabla_j \nabla_i \psi_a = -\frac{1}{4} \sigma^{pq}{}_{,a}{}^b \psi_b R_{jipq}, \quad (7)$$

where R_{jipq} is the curvature tensor of the space G .

2. Let (x^i) be the reference frame of an observer studying physical phenomena in the medium under consideration, and let (ξ^i) be a comoving coordinate system: ξ^α ($\alpha = 1, 2, 3$) determine the individual points of the medium—the continuum; variation of ξ^4 at fixed ξ^α gives the world line of a point of the continuum in the space G . By the law of motion of the medium we shall mean the relation $x^i = x^i(\xi^k)$.

Consider the general variational equation of the form ^(3, 4)

$$\delta \int_V L d\tau + \delta W + \delta W^* = 0. \quad (8)$$

where

$$L = \Lambda(\psi_a, \nabla_i \psi_a, \nabla_s \nabla_j \psi_a, \psi_a^+, \gamma_{,b}^{i a}, e^{ab}, A_i, \nabla_k A_i, x_p^i, S, l^k) + \frac{1}{2\chi} R + \frac{1}{16\pi} F_{kl} F^{kl}, \quad (9)$$

Λ is a prescribed scalar function of the indicated arguments; ψ_a is a certain spinor field; R is the scalar curvature of G ; $F_{kl} = \nabla_k A_l - \nabla_l A_k$ is the electromagnetic-field tensor; A_i is the electromagnetic potential; $x_p^i = \partial x^i / \partial \xi^p$; the scalar S is entropy; l^k are prescribed nonvaried tensors with components in the system (ξ^i) ; V is an arbitrary four-dimensional volume; $d\tau$ is the invariant element of this volume. The variation δW is represented by an integral over the surface Σ bounding V , of a linear combination of the variations δg^{ij} , $\nabla_k \delta g^{ij}$, $\delta \psi_a$, $\delta \nabla_k \psi_a$, δx^i , δA_i and is determined by the specification of the function Λ and of δW^* , which is defined by the formula

$$\begin{aligned} \delta W^* = \int_V [\rho \theta \delta S - Y^a \delta \psi_a - X^k \delta \psi_k^+ - Q_i \delta x^i - N^l \delta A_l \\ - \nabla_j (Y^{j,a} \delta \psi_a + Y^{jk,a} \delta \nabla_k \psi_a + Q_i^j \delta x^i + N^{jl} \delta A_l)] d\tau, \end{aligned} \quad (10)$$

where all coefficients multiplying the variations are prescribed, and the scalar ρ is the density, defined by the relation $\nabla_i (\rho u^i) = 0$, representing the continuity equation, where $u^i = dx^i / ds$ is the four-velocity, ds is the element of length along the world line. All functions are assumed sufficiently smooth. The symbol δ denotes variation at constant ξ^i of the form:

$$\delta \mu = \partial \mu + \delta x^i \nabla_i \mu, \quad (11)$$

where $\partial \mu = \mu'(x) - \mu(x)$, and μ is an arbitrary spintensor. The spintensors $\gamma_{,b}^{i a}$, e^{ab} are not varied.

Let us write down several auxiliary formulas that can be obtained from relations of the form (11):

$$\delta x_p^i = x_p^s \nabla_s \delta x^i,$$

$$\delta \nabla_j \psi_a = \nabla_j \delta \psi_a + \delta x^i (\nabla_i \nabla_j \psi_a - \nabla_j \nabla_i \psi_a) - \nabla_j \delta x^i \cdot \nabla_i \psi_a - \psi_b \partial \Gamma_{j,a}^b,$$

$$\delta \nabla_j A_k = \nabla_j \delta A_k + \delta x^i (\nabla_i \nabla_j A_k - \nabla_j \nabla_i A_k) - \nabla_j \delta x^i \cdot \nabla_i A_k - A_s \partial \Gamma_{jk}^s, \quad (12)$$

$$\delta \nabla_i \nabla_j \psi_a = \nabla_i \delta \nabla_j \psi_a - \nabla_i \delta x^s \cdot \nabla_s \nabla_j \psi_a + \delta x^p (\nabla_p \nabla_i \nabla_j \psi_a - \nabla_i \nabla_p \nabla_j \psi_a) - \nabla_j \psi_b \partial \Gamma_{i,a}^b - \nabla_p \psi_a \partial \Gamma_{ij}^p,$$

where

$$\partial \Gamma_{i,b}^a = A_{ij,b}^k \nabla_k \partial g^{il}, \quad \partial \Gamma_{jk}^s = B^{sp}_{jkil} \nabla_p \partial g^{il},$$

$$A^i_{ksq,b}{}^a = \frac{1}{8} (g_{rs} \delta_q^p + g_{kq} \delta_s^p) \sigma_p{}^{ia}{}_{,b},$$

$$B^{nk}_{lmij} = \frac{1}{4} [g^{nk} (g_{il} g_{jm} + g_{im} g_{jl}) - \delta_j^n (g_{li} \delta_m^k + g_{mi} \delta_l^k) - \delta_i^n (g_{lj} \delta_m^k + g_{mj} \delta_l^k)]. \quad (13)$$

From (2) and (6) we have

$$\delta \gamma^i{}_{,b}{}^a = \partial \gamma^i{}_{,b}{}^a = \frac{1}{2} \gamma_{s,b}{}^a \partial g^{is}.$$

Substituting into (8) (see (4)), taking these formulas into account and applying the generalized Stokes theorem, in view of the arbitrariness of the variations ∂g^{il} , $\delta \psi_a$, $\delta \psi_a^+$, δx^i , δA_l , δS , and putting δW and all variations on Σ equal to zero, we obtain

$$\frac{1}{2} \left(\frac{\partial \Lambda}{\partial \gamma^i{}_{,b}{}^a} \gamma_{l,b}{}^a + \frac{\partial \Lambda}{\partial \gamma^l{}_{,b}{}^a} \gamma_{i,b}{}^a \right) + \frac{1}{\chi} R_{il} + \frac{1}{4\pi} F^s{}_i F_{ls} - L g_{il} + 2 \nabla_k D^k{}_{.il} = 0; \quad (14)$$

$$\frac{1}{2\chi} \nabla_i R - \frac{1}{4} \left[\left(\frac{\partial \Lambda}{\partial \nabla_j \psi_a} - \nabla_s \frac{\partial \Lambda}{\partial \nabla_s \nabla_j \psi_a} \right) \sigma^{pq}{}_{,a}{}^b \psi_b + \frac{\partial \Lambda}{\partial \nabla_j \nabla_s \psi_a} \sigma^{pq}{}_{,a}{}^b \nabla_s \psi_b \right] R_{jipq} - \left(\frac{\partial \Lambda}{\partial \nabla_j A_k} + \frac{1}{4\pi} F^{jk} \right) R^s{}_{jik} A$$

$$\partial \Lambda / \partial \psi_a + \nabla_k \Phi^k{}_{,a} = Y^a + \nabla_k Y^{k,a}; \quad (16)$$

$$\partial \Lambda / \partial \psi_a^+ = X^a; \quad (17)$$

$$\partial \Lambda / \partial A_i + \nabla_j M^{ji} = N^i + \nabla_j N^{ji}; \quad (18)$$

$$\partial\Lambda/\partial S = -\rho\theta, \quad (19)$$

where

$$R_{il} = R_{ise}{}^s;$$

$$D^k{}_{.il} = \left(\frac{\partial\Lambda}{\partial\nabla_s\psi_a} - \nabla_j \frac{\partial\Lambda}{\partial\nabla_j\nabla_s\psi_a} \right) A^k{}_{.sil,a}{}^b\psi_b + \frac{\partial\Lambda}{\partial\nabla_s\nabla_j\psi_a} (A^k{}_{.sil,a}{}^b\nabla_j\psi_b + B^{pk}{}_{.sjil}\nabla_p\psi_a) + \frac{\partial\Lambda}{\partial\nabla_p A_j} B^{sk}{}_{.pjil} A_s$$

$$P^k{}_{.i} = \left(\frac{\partial\Lambda}{\partial\nabla_k\psi_a} - \nabla_j \frac{\partial\Lambda}{\partial\nabla_j\nabla_k\psi_a} \right) \nabla_i\psi_a + \frac{\partial\Lambda}{\partial\nabla_k\nabla_j\psi_a} \nabla_i\nabla_j\psi_a + \left(\frac{\partial\Lambda}{\partial\nabla_k A_l} + \frac{1}{4\pi} F^{kl} \right) \nabla_i A_l - x_p^k \frac{\partial\Lambda}{\partial x_p^i} - L\delta_i^k + Q^k{}_{.i};$$

$$\Phi^{ka} = -\frac{\partial\Lambda}{\partial\nabla_k\psi_a} + \nabla_s \frac{\partial\Lambda}{\partial\nabla_s\nabla_k\psi_a} + Y^{ka}, \quad M^{kl} = -\frac{\partial\Lambda}{\partial\nabla_k A_l} - \frac{1}{4\pi} F^{kl} + N^{kl}.$$

Taking equations (14)–(19) into account, from (8) we obtain

$$\delta W = \int_{\Sigma} (D^k{}_{.il}\delta g^{il} + G^{kp}{}_{.il}\nabla_p\delta g^{il} + \Phi^k{}_{,a}\delta\psi_a + \Phi^{ks}{}_{,a}\delta\nabla_s\psi_a + P^k{}_{.j}\delta x^j + M^{kl}\delta A_l) n_k d\sigma, \quad (21)$$

where

$$G^{ks}{}_{.il} = \frac{1}{4\chi} (\delta_l^k\delta_i^s + \delta_i^k\delta_l^s - 2g^{ks}g_{il}); \quad \Phi^{ks}{}_{,a} = -\frac{\partial\Lambda}{\partial\nabla_k\nabla_s\psi_a} + Y^{ks}{}_{,a}. \quad (20')$$

3. The definitions of the coefficients in δW may be regarded as equations of state of the medium, determining the internal generalized stresses. Equations (14)–(16), (18) are certain laws of change of the quantities $D^k{}_{.il}$, $P^k{}_{.j}$, $\Phi^k{}_{,a}$, M^{ji} , which characterize the generalized stresses acting on the variations δg^{il} , δx^i , $\delta\psi_a$, δA_i . The spinors Y^a , X^a and the vectors Q_j , N^i enter equations (15)–(18) as generalized forces. The coefficients of the variations in δW correspond to volume and surface sources of energy.

Consider a variation caused only by a rotation of the local frame $\partial\omega^{ij} = -\partial\omega^{ji}$, and let $\nabla_s\partial\omega^{ij}$ be set equal to zero; then from (4)

$$\delta\psi_a = \partial\psi_a = \frac{1}{4}\sigma_{ij,a}{}^b\psi_b\partial\omega^{ij}, \quad \delta\nabla_s\psi_a = \frac{1}{4}\sigma_{ij,a}{}^b\nabla_s\psi_b\partial\omega^{ij} \quad \text{and} \quad \delta W = \int_{\Sigma} S^k{}_{.ij}\partial\omega^{ij}n_k d\sigma,$$

where

$$S^k{}_{.ij} = -\frac{1}{4} (\Phi^{ka} \sigma_{ij,a}{}^b \psi_b + \Phi^{ks,a} \sigma_{ij,a}{}^b \nabla_s \psi_b). \quad (22)$$

Equality (22) determines the distribution of internal spin moments, which include the spin-tensors $Y^{k,a}$ and $Y^{ks,a}$. Equations (18) are the second pair of Maxwell equations.

The tensor $P^k{}_{.j}$ is called the energy-momentum tensor. Taking (16)–(19) into account, the equation of motion of the medium (15) takes the form

$$\nabla_k \left(x_p^k \frac{\partial \Lambda}{\partial x_p^i} + \frac{\partial \Lambda}{\partial x_p^k} \nabla_i x_p^k - \rho \theta \nabla_i S + \frac{\partial \Lambda}{\partial l^k} \frac{\partial l^k}{\partial \xi^p} \frac{\partial \xi^p}{\partial x^i} + Q_i + N^l \nabla_i A_l + Y^a \nabla_i \psi_a + X^a \nabla_i \psi_a^+ \right) = 0^*. \quad (23)$$

In the absence of a medium, i.e. if the arguments of Λ do not include S , x_p^i , and l^k , and with θ , Q_j , N^i , Y^a , and X^a in δW^* equal to zero, equation (15) is a consequence of equations (16)–(18). In this case $P^k{}_{.j}$ need not be introduced, nor need x^i be varied. The tensor T_{il} , appearing in Einstein's equation (14),

$$R_{il} - \frac{1}{2} R g_{il} + \chi T_{il} = 0,$$

$$T_{il} = \frac{1}{2} \left(\frac{\partial \Lambda}{\partial \gamma^i{}_{,b}{}^a} Y^a{}_{l,b} + \frac{\partial \Lambda}{\partial \gamma^l{}_{,b}{}^a} Y^a{}_{i,b} \right) + \frac{1}{4\pi} \left(F_i{}^s F_{ls} - \frac{1}{4} F_{ks} F^{ks} g_{il} \right) - \Lambda g_{il} + 2 \nabla_k D^k{}_{.il}$$

in the general case does not coincide with the tensor P_{ki} .

The energy equation is obtained from (23) by contracting with u^i , and it may be regarded as the entropy-balance equation for $\theta \neq 0$

$$\rho DS = \frac{1}{\theta} \left[Q_i u^i + N^l D A_l + Y^a D \psi_a + X^a D \psi_a^+ + \frac{1}{\sqrt{\hat{g}^{44}}} \left(\nabla_k F^k + \frac{\partial \Lambda}{\partial l^k} \frac{\partial l^k}{\partial \xi^4} \right) \right], \quad (24)$$

where $F^k = \frac{\partial \Lambda}{\partial x_p^i} x_p^k x_4^i$, the operator $D = u^i \nabla_i$, and $\hat{g}_{ij}$ is the metric tensor in the system (ξ^k) . If one uses a more concrete specification of the functions Λ , θ , Y^a , X^a , Q_j , N^i , and relies on their physical meaning, then equation (24) can be written in the form $\rho DS = \nabla_k \hat{H}^k + \sigma$, where $\hat{H}^4 = 0$ in the proper reference frame,** and, according to the second law of thermodynamics, $\sigma \geq 0$. This condition imposes restrictions on the specification of Λ , θ , Y^a , X^a , Q_j , and N^i .

The formalism developed above makes it possible to introduce, as defining parameters characterizing the medium, a spinor field with its first and second covariant derivatives, which is determined by equations (16) and (17). The introduction of δW^* describes irreversible processes occurring in the medium. In addition, in the absence of a medium one can obtain a generalization of the Dirac equation and of the Klein-Gordon equation in elementary-particle theory to the case of their interaction with gravitational and electromagnetic fields.

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* In the expression $\nabla_i x_p^k$, covariant differentiation acts only on the index k .

** A nonholonomic coordinate system $(\tilde{x}^i)$ with the axis $\tilde{x}^4$ directed along the tangent to the world line.

Note: Figure translations are in progress. See original paper for figures.

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