

GROUPS WITH PRESCRIBED PROPERTIES OF SYSTEMS OF INFINITE SUBGROUPS

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Abstract

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MATHEMATICS

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GROUPS WITH PRESCRIBED PROPERTIES OF SYSTEMS OF INFINITE SUBGROUPS

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The study of groups in which certain subgroups, or systems of subgroups, satisfy one or another requirement has long played an essential role in the development of the general theory of groups and has led to the isolation of many important classes of groups. It has also yielded a number of substantial results describing the structure of various kinds of groups with restrictive requirements on subgroups. Among the first to be studied were nonabelian groups all of whose subgroups are invariant (normal divisors)—Hamiltonian groups (H -groups), which turned out to be periodic groups very close to abelian groups (see ⁽¹⁾). A more general class of groups is formed by the groups all of whose proper subgroups differ from their normalizers— N -groups, whose study was begun later. In the definition of N -groups the restrictive requirement is imposed, generally speaking, both on finite and on infinite subgroups. In the case of infinite groups this requirement can be weakened, retaining it only for infinite subgroups. Since not every infinite group has infinite proper subgroups, the corresponding definition should be formulated in the following form. An infinite group having no infinite proper subgroup that coincides with its normalizer will be called a JN -group. In particular, every infinite group that has no invariant infinite subgroups is a JN -group. Since in group theory the question of the existence of infinite nonabelian groups all of whose proper subgroups are finite has not yet been solved, the study of JN -groups in the general case, in the absence of restrictions excluding this case of JN -groups, cannot at present reveal their structure. In the present article such a restriction is local finiteness.

It is clear that this restriction is also suitable in studying a generalization of H -groups—namely, infinite nonabelian groups having no noninvariant infinite subgroups; however, in this case one can use a weaker restriction, the requirement of the existence of an infinite abelian subgroup. Incidentally, if Hamiltonian groups are defined as nonabelian groups with invariant abelian subgroups (with invariant cyclic subgroups), then it is appropriate to study the more general class of infinite groups with the following definition. An infinite nonabelian group having at least one infinite abelian subgroup will be called a JH -group if all its infinite abelian subgroups are invariant in it.

The results set out below, obtained by the author in the investigation of JH - and JN -groups, rely substantially on the main results of his works ^(2, 3). The theorem below concerning JN -groups (Theorem 5) deepens and completes the corresponding proposition on JN -groups from the author's note ⁽⁴⁾.

1. Let us first note the case of a JH -group containing elements of infinite order. It is not difficult to verify the validity of the following proposition.

A nonabelian group $\mathfrak{G}$ containing elements of infinite order is a JH -group if and only if it has an abelian normal divisor $\mathfrak{A}$ of index 2 and an element B of second or fourth order such that $\mathfrak{G} = \mathfrak{A}\{B\}$, where $\{B\}$ is the cyclic subgroup generated by the element B , and $B^{-1}AB = A^{-1}$ for every element $A \in \mathfrak{A}$.

This proposition shows, in particular, that JH -groups containing elements of infinite order are necessarily mixed groups, i.e., groups containing, along with elements of infinite order, also nonidentity elements of finite order. Using it, it is not difficult to verify that the center of a JH -group containing elements of infinite order is finite and consists only of elements of order not exceeding two.

For periodic JH -groups the following assertion is valid.

Every periodic JH -group is locally finite and locally soluble.

Using this assertion and the main results of the author's paper ⁽²⁾, one can obtain the following key theorem for the study of the structure of periodic JH -groups.

Theorem 1. Every periodic JH -group $\mathfrak{G}$ contains an abelian normal divisor of finite index.

In view of the already noted local solubility of periodic JH -groups, it follows from this that every periodic JH -group is soluble. The solubility of JH -groups containing elements of infinite order follows from the proposition formulated at the beginning of the present section.

2. Hamiltonian groups are periodic JH -groups. However, there also exist non-Hamiltonian periodic JH -groups. In their construction an essential role is played by the following simplest JH -groups.

Nonabelian p -groups (p any prime number) that are central extensions of a quasicyclic group by finite abelian groups; we shall call them **quasifinite JH -groups of the first kind**.

Groups that are central extensions of a quasicyclic 2-group by finite Hamiltonian 2-groups; we shall call them **quasifinite JH -groups of the second kind**.

These groups are the simplest nonabelian infinite layer-finite p -groups; their complete description can be obtained by using the results of the author's work ⁽⁵⁾.

Theorem 2. A periodic JH -group $\mathfrak{G}$ with infinite center is non-Hamiltonian if and only if it decomposes into the direct product of a quasifinite JH -group

of the first kind with odd p , which is its Sylow subgroup, and a finite abelian or finite Hamiltonian group, or into the direct product of a quasifinite JH -group of the first or second kind, which is a Sylow 2-subgroup of the group $\mathfrak{G}$, and a finite abelian group.

From Theorems 1 and 2 it follows.

Corollary 1. For odd p , an infinite p -group $\mathfrak{P}$ having infinite abelian subgroups but having no noninvariant subgroups of this kind is nonabelian if and only if it is a quasifinite JH -group of the first kind.

Corollary 2. For $p = 2$, an infinite nonabelian p -group $\mathfrak{P}$ with infinite center and invariant infinite abelian subgroups is non-Hamiltonian if and only if it is a quasifinite JH -group of the first or second kind.

3. **Theorem 3.** Every periodic JH -group $\mathfrak{G}$ with finite center has an invariant abelian subgroup $\mathfrak{A}$, or such an invariant JH -subgroup $\mathfrak{A}$ with infinite center, which determines

a nontrivial cyclic factor group $\mathfrak{G}/\mathfrak{N}$ and satisfies one of the following conditions:

- 1) if the group $\mathfrak{G}$ is not a finite extension of a quasicyclic group, then all cyclic subgroups of $\mathfrak{N}$ are invariant in $\mathfrak{G}$;
- 2) if the group $\mathfrak{G}$ is a finite extension of a quasicyclic subgroup $\mathfrak{K}$, then all cyclic subgroups of $\mathfrak{N}/\mathfrak{K}$ are invariant in $\mathfrak{G}/\mathfrak{K}$.

In a JH -group $\mathfrak{G}$ with finite center there exists an invariant subgroup $\mathfrak{N}$ with the properties described here such that there is no subgroup $\mathfrak{N}' \subset \mathfrak{G}$, containing it and distinct from it, with infinite center.

Corollary 3. If a periodic JH -group $\mathfrak{G}$ with finite center is not a finite extension of a quasicyclic group, then it contains such an abelian or Hamiltonian normal divisor $\mathfrak{N}$ which determines a nontrivial cyclic factor group $\mathfrak{G}/\mathfrak{N}$.

In addition to Theorem 3 we note the following strengthening of it, concerning 2-groups.

Corollary 4. In order that an infinite 2-group $\mathfrak{P}$ with finite center, which is a finite extension of a quasicyclic group $\mathfrak{K}$, be a JH -group, it is necessary and sufficient that it contain such a normal divisor $\mathfrak{N}$ of index 2, which is either an abelian group or a quasifinite JH -group of the first or second kind, that all cyclic subgroups of $\mathfrak{N}/\mathfrak{K}$ be invariant in $\mathfrak{G}/\mathfrak{K}$.

Theorem 3*. An infinite periodic group $\mathfrak{G}$ with finite center is a JH -group if it possesses such an invariant subgroup $\mathfrak{N}$ with cyclic factor group $\mathfrak{G}/\mathfrak{N}$ and infinite center, that the following conditions are fulfilled:

- 1) there is no (invariant) subgroup with infinite center containing $\mathfrak{N}$ and distinct from $\mathfrak{N}$;
- 2) all cyclic subgroups of $\mathfrak{N}$ are invariant in the group $\mathfrak{G}$, if the latter is not a finite extension of a quasicyclic group, or all cyclic subgroups of $\mathfrak{N}/\mathfrak{K}$

are invariant in $\mathfrak{G}/\mathfrak{K}$, if the group $\mathfrak{G}$ is a finite extension of a quasicyclic group $\mathfrak{K}$.

4. Let us also consider JH -groups in which all infinite subgroups are invariant; we shall call them JHH -groups. It is not difficult to see that all JHH -groups are periodic. With the aid of the propositions given above, describing the structure of periodic JH -groups, one can verify the validity of the following proposition.

Theorem 4. In order that a (periodic) JH -group $\mathfrak{G}$ be a JHH -group, it is necessary and sufficient that one of the following two conditions be fulfilled:

- 1) the center of the group $\mathfrak{G}$ is infinite;
- 2) the center of the group $\mathfrak{G}$ is finite and it is an extension of a quasicyclic group by means of a finite abelian or finite Hamiltonian group.

Corollary 5. A non-Hamiltonian JHH -group is an extension of a quasicyclic group by means of a finite abelian or finite Hamiltonian group.

Since every infinite locally finite group has infinite abelian subgroups (see (5)), it follows from this that

Corollary 6. An infinite nonabelian locally finite group with invariant infinite subgroups is either Hamiltonian, or an extension of a quasicyclic group by means of a finite abelian or finite Hamiltonian group.

5. The present section is devoted to locally finite JN -groups. In order to formulate a theorem describing the structure of such groups, we give the following definition. A nonidentity automorphism φ of the direct product $\mathfrak{N}$ of a finite number of quasicyclic p -groups with one

call it, and the same p , **irreducible** if this product contains no infinite true complete subgroups admissible with respect to φ .

Theorem 5. If an infinite locally finite JN -group does not satisfy the normalizer condition, then it decomposes into a semidirect product of an infinite invariant Sylow p -subgroup $\mathfrak{P}$ and a finite nilpotent subgroup $\mathfrak{S}$, satisfying the following conditions:

- 1) the subgroup $\mathfrak{P}$ is a finite extension of a direct product $\mathfrak{N}$ of a finite number of quasicyclic groups;
- 2) the factor group $\mathfrak{S}/\mathfrak{N}$ is nilpotent;
- 3) the factor group $\mathfrak{S}/\mathfrak{S}^*$, where $\mathfrak{S}^*$ is the subgroup of elements of $\mathfrak{S}$ commuting with the elements of the group $\mathfrak{N}$, is a cyclic group different from the identity;
- 4) all nonidentity elements of the group $\mathfrak{S}/\mathfrak{S}^*$ induce in the subgroup $\mathfrak{N}$ mutually distinct irreducible automorphisms.

Every infinite (locally finite) group $\mathfrak{G}$ satisfying all the requirements stated here is a JN -group not satisfying the normalizer condition.

Using condition 2), it is not hard to verify that conditions 3) and 4) hold for any complement of the subgroup $\mathfrak{P}$ in $\mathfrak{G}$.

In the proof of Theorem 5 the following property of periodic subgroups of irreducible automorphisms of the direct product $\mathfrak{R}$ of a finite number of quasicyclic p -groups with one and the same p is used essentially; for brevity, we shall call a subgroup of the automorphism group of the direct product $\mathfrak{R}$ **rigid** if it is different from the identity and all its elements different from the identity are irreducible.

Theorem 6. A periodic rigid subgroup $\mathfrak{H}$ of the automorphism group of a direct product $\mathfrak{R}$ of a finite number of quasicyclic p -subgroups with one and the same p is a finite group with cyclic Sylow subgroups. The least (prime) divisor of the order of the group $\mathfrak{H}$ exceeds the number of factors of the product $\mathfrak{R}$.

Remark. A finite group with cyclic Sylow subgroups is either cyclic, or a semidirect product of two cyclic groups of coprime orders (see ⁽¹⁾, Sec. 9.4).

In the proof of Theorem 6 the following proposition is used, which is also of some independent interest.

Let the group $\mathfrak{G}$ be a semidirect product of a p -group $\mathfrak{R}$, decomposing into a direct product of a finite number of quasicyclic groups, and a finite noncyclic elementary abelian q -group $\mathfrak{Q}$, with $q \neq p$. Then the centralizer in $\mathfrak{G}$ of at least one element different from the identity in $\mathfrak{Q}$ is infinite.

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