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Abstract

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MATHEMATICS

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ON THE GLOBAL BEHAVIOR OF PHASE TRAJECTORIES OF QUASILINEAR DIFFERENTIAL EQUATIONS WITH DELAYED ARGUMENTS

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The paper considers the global behavior of phase trajectories of a dynamical system described by quasilinear differential equations with delayed arguments of the form*

$$\dot{x} = A_0x + A_1x(t - \tau_1) + \dots + A_mx(t - \tau_m) + \mu f(t; x(t), x(t - \tau_1), \dots, x(t - \tau_m)). \quad (1)$$

Here x is an n -dimensional vector; $A_0, A_1, \dots, A_m$ are constant matrices; $0 < \tau_1 < \dots < \tau_m < 2\pi$ are constant delays; μ is a small parameter; $f(t; u_0, u_1, \dots, u_m)$ is a function of $t, u_0, u_1, \dots, u_m$, continuous together with its partial derivatives with respect to $u_0, u_1, \dots, u_m$, and periodic in the variable t with period 2π .

For the system of quasilinear equations (1), as for ordinary quasilinear differential equations, the case in which the corresponding linear system admits a family of harmonic solutions is of particular interest. It turns out that if $p_1, p_2, \dots, p_s$ are purely imaginary roots of the quasi-polynomial

$$\text{Det } \chi(p) = \text{Det}\{pE - A_0 - A_1e^{-p\tau_1} - \dots - A_me^{-p\tau_m}\}, \quad (2)$$

and the remaining roots have negative real parts, then, under certain assumptions that are sufficiently broad from the applied point of view, in any bounded domain $\|x\| < K$, for sufficiently small μ , the asymptotic behavior of phase trajectories as $t \rightarrow +\infty$ is described, up to quantities of order μ , by a system of ordinary differential equations of order s , obtained in a definite way from equations (1).**

1. Any particular solution of equation (1) for $t \geq t_0$ is uniquely determined by specifying the function $x(t)$ on the interval from $t_0 - \tau_m$ to t_0 . In view of this, the phase space Φ of the dynamical system described by equation (1) may be regarded as the space of continuous functions $x(t)$ defined on the intervals $[t_0 - \tau_m, t_0]$. In the case of an autonomous system, the state of the system is then determined by prescribing $x(t)$ on an interval of length τ_m , independently of t_0 , i.e. one may assume that the state of the system is determined by prescribing the function $\varphi(\tau) = x(t_0 + \tau)$ for $-\tau_m \leq \tau \leq 0$. For a nonautonomous system, the value of t_0 is important, so that the state of a nonautonomous system may be defined by means of a function $\varphi(\tau)$, prescribed on the fixed interval $[-\tau_m, 0]$, and the time instant t_0 ,

* Numerous works are devoted to the study of differential equations with delayed arguments (see the detailed surveys ⁽¹⁻³⁾). Quasilinear equations with delay, in particular, were studied in works ⁽⁴⁻⁹⁾.

** We note that in the presence of a root with positive real part, almost all phase trajectories leave the domain $\|x\| < K$ for sufficiently small μ .

which, by virtue of the periodicity of the first parts of equations (1) with respect to t , may be regarded as lying between 0 and 2π .

In the introduced phase space Φ , define the distance between any two phase points $\xi^1(\varphi^1(\tau), t_0^1)$ and $\xi^2(\varphi^2(\tau), t_0^2)$ by

$$\rho(\xi_1, \xi_2) = \sup_{-\tau_m \leq \tau \leq 0} |\varphi^1(\tau) - \varphi^2(\tau)| + \min \{|t_0^1 - t_0^2|, 2\pi - |t_0^1 - t_0^2|\} \quad (3)$$

and in it the point mapping $T_{2\pi}$, which assigns to each phase point $\xi(\varphi(\tau), t_0)$ the point $\xi(\bar{\varphi}(\tau), t_0)$ into which it passes, according to equations (1), in time 2π . The mapping $T_{2\pi}$ is, obviously, completely continuous.

In what follows we consider the phase trajectories of the dynamical system (1) in the bounded part $\|\xi\| < K$ of the introduced phase space, where, with respect to the number K , it is assumed only that it is some fixed number.

2. Let us first consider the special case when $A_1 = A_2 = \dots = A_m = 0$. In this case, for $\mu = 0$, the system of equations (1) becomes a system of ordinary differential equations, and the corresponding point mapping $T_{2\pi}$ "degenerates" into the mapping of the finite-dimensional phase space of the corresponding system of linear differential equations. For $\mu \neq 0$, associate with the system of equations (1) the system of ordinary differential equations

$$\dot{x} = A_0 x + \mu f(t; x, e^{-A_0 \tau_1} x, \dots, e^{-A_0 \tau_m} x). \quad (4)$$

The corresponding mapping $T_{2\pi}$ of the $(n+1)$ -dimensional phase space can be extended in a natural way to the space Φ . Namely, define the extension $T_{2\pi}$, which we denote by $\tilde{T}_{2\pi}$, as the mapping which assigns to the point $\xi(\varphi(\tau), t_0)$ the point $\xi(x(t_0 + 2\pi + \tau), t_0)$, where $x(t)$ is the solution of equation (4) that becomes $\varphi(0)$ at $t = t_0$. Denote by S the ball $\|x\| < K$ ($0 < K < \infty$) in the space Φ .

The following almost obvious lemma holds.

Lemma 1. *On the set S , and correspondingly $T_{2\pi}S$, for sufficiently small μ and certain constants K_1 and K_2 ,*

$$\|T_{2\pi}\xi - (T_{2\pi})_{\mu=0}\xi\| < \mu K_1, \quad (5)$$

$$\|T_{2\pi}\xi - \tilde{T}_{2\pi}\xi\| < \mu^2 K. \quad (6)$$

Indeed, the solution of equation (1) in the special case under consideration ($A_1 = A_2 = \dots = A_m = 0$) satisfies the equation

$$x(t) = e^{A_0(t-t_0)}\varphi(0) + \mu \int_0^t e^{A(t-t_0-v)} f(v, x(v), \dots, x(v - \tau_m)) dv, \quad (7)$$

and therefore, for sufficiently small μ , from $\|\varphi(\tau)\| < K$ it follows that, for some K' , $\|x(t)\| < K'$ for all $t_0 \leq t \leq t_0 + 2\pi$; whence, in turn, according to (7), the estimate (5) follows. Now, according to estimate (5) and equation (7), we immediately obtain (6).

The behavior of the phase trajectories of the system of differential equations with delays (1) and of the system of ordinary differential equations (4) is described by the corresponding point mappings $T_{2\pi}$ of the sections $t_0 = 0$ of the phase spaces of these systems. Suppose that the point mapping $T_{2\pi}$ of the dynamical system (4) has an attracting element J (a stable equilibrium point, a stable invariant curve, etc.) with domain of attraction Π , and suppose that in a neighborhood of J

$$\rho(T_{2\pi}x, J) < (1 - q\mu)\rho(x, J) + M\mu^2 \quad (q > 0, M < +\infty). \quad (8)$$

According to the lemma of Section 3 of work (10), for any ε and any bounded closed domain Π strictly interior to P , one can indicate an $N(\varepsilon)$ such that, for any point $x \in \Pi'$, for some $j < N\mu^{-1}$ the point $T_{2\pi}^j x$ belongs to the ε -neighborhood of J . Denote by $\tilde{\mathcal{T}}$ and $\tilde{\Pi}$ the sets consisting of the points $\xi(\varphi(\tau), 0) \in T_{2\pi}S$ for which $\varphi(0) \in J$ or $\varphi(0) \subset \Pi$, respectively. For the point mapping $\tilde{T}_{2\pi}$ and the sets $\tilde{\mathcal{T}}$ and $\tilde{\Pi}$, by virtue of Lemma 1, relation (8) holds, and the following theorem holds:

Theorem 1. *Under the assumptions formulated above concerning the behavior of the phase trajectories of system (4), for sufficiently small μ , all points of any compact strictly interior part of the domain $\tilde{\Pi}$ with $A_1 = A_2 = \dots = A_m = 0$, according to equation (1), arrive and remain, as $t \rightarrow +\infty$, in an ε -neighborhood of $\tilde{J}$, where ε is of order μ .*

3. Applying the Laplace transform to equation (1), in the usual way we find the image $x(p)$ of the solution $x(t)$ in the form

$$x(p) = \chi^{-1}(p)\{\mu F(p) + \Phi(p)\}, \quad (9)$$

where $\chi(p)$ is the matrix whose determinant follows from (2),

$$\Phi(p) = -\varphi(0) - A_1 \int_{-\tau_1}^0 \varphi(\nu) e^{-p(\nu+\tau_1)} d\nu - \dots - A_m \int_{-\tau_m}^0 \varphi(\nu) e^{-p(\nu+\tau_m)} d\nu \quad (10)$$

and $F(p)$ is the image of $f(t; x(t), x(t - \tau_1), \dots, x(t - \tau_m))$.

The inverse of the matrix $\chi(p)$ can be represented in the form

$$\chi^{-1}(p) = \sum_{k=1}^s \frac{\psi(p_k)}{p - p_k} + \Omega(p), \quad (11)$$

where the matrices $\psi(p_k)$ are of unit rank, and the poles of the function $\Omega(p)$ lie in the half-plane $\operatorname{Re} p < -\sigma < 0$. In accordance with (11), the solution $x(t)$ can be written in the form

$$x(t) = \sum \xi_k(t) + \eta(t), \quad (12)$$

where $\xi_k(t)$ satisfy equations of the form*

$$\dot{\xi}_k = p_k \xi_k + \mu \psi(p_k) f\left(t, \sum \xi_k(t) + \eta(t), \dots, \sum \xi_k(t - \tau_m) + \eta(t - \tau_m)\right) \quad (13)$$

$$(k = 1, 2, \dots, s),$$

and, for some constants A and B ,

$$\|\eta(t)\| < Ae^{-\sigma t} + \mu B. \quad (14)$$

Consequently, up to quantities of order μ , the study of the asymptotic behavior of solutions of equation (1) reduces to the study of solutions of equation (13).

Equation (13), if we put $\eta(t) = 0$ in it, which introduces into the corresponding mapping $T_{2\pi}$ an error of order μ^2 , belongs to the special case considered above.

Let us summarize the foregoing for the practically frequently occurring special case when $s = 2$ and $p_1 = i\omega$, $p_2 = -i\omega$, and when the averaged equation corresponding to equation (13) is rough in the sense of Andronov–Pontryagin.

Theorem 2. *In the stated case, for sufficiently small μ , the behavior of the phase trajectories of system (1), up to quantities of order μ , reduces to the study of a system of autonomous equations of second order, obtained by averaging equations (13), in which $\eta(t) = 0$ is put and the quantities $\xi_k(t - \tau_j)$ are replaced by $e^{-p_k \tau_j} \xi_k(t)$.*

* Among the ns equations (13), since the ranks of the matrices $\psi(p_k)$ are equal to one, only s are independent.

This theorem may be regarded as a generalization of the averaging method to systems of equations of the form (1). We note that in the special case when $A_1 = A_2 = \dots A_m = 0$, some results on the justification of the averaging method were obtained in works (7–9).

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