

LIMITS OF POSITIVE MOLECULAR MEASURES IN CERTAIN UNIFORM SPACES

I. A. Berezanskii

1966

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196601.05940>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.881

MATHEMATICS

I. A. BEREZANSKII

LIMITS OF POSITIVE MOLECULAR MEASURES IN CERTAIN UNIFORM SPACES

(Presented by Academician P. S. Novikov on 4 IV 1966)

In the author's preceding note ⁽⁴⁾, the dual scheme of Katětov, Raikov, and Ptak ⁽¹⁻³⁾ was applied to the problem of approximating bounded Borel measures on a topological or uniform space by means of molecular measures, i.e., measures concentrated at a finite number of points. In addition, the question was studied of the extent to which all objects obtained with the aid of the approximative process (completion, bounded completion) can be identified with measures. In the present note the study of these questions is continued for a broader class of uniform spaces, but only as applied to signed-positive measures.

The basic definitions are given in ⁽⁴⁾. The field of scalars in what follows will always be R . For a given element

$$u = \sum_{x \in X} \lambda_x x \in L(X)$$

we shall denote by N_u the set $\{x \in X : \lambda_x \neq 0\}$. Let (X, ξ) be a separated uniform space. As the space $P(X)$ we shall take the set of all bounded uniformly continuous real functions on (X, ξ) , and as $\mathcal{H}$ the family of all uniformly bounded and equicontinuously uniformly continuous subsets of $P(X)$. If $T \subset X$, then by $u[T]$ we shall denote

$$\sum_{x \in T} \lambda_x.$$

Lemma 1. Let $\{u_\alpha\}_{\alpha \in A}$ be a Cauchy net of positive elements of $(L(X), \tau_H)$. In that case, for every $\varepsilon > 0$ and for every entourage $V \in \xi$, there exists an index $\alpha_0 \in A$ such that $u_\alpha[X \setminus V(N_{u_{\alpha_0}})] < \varepsilon$ for all $\alpha \geq \alpha_0$.

Lemma 2. Let $x_{\alpha\beta}$ ($\alpha \in A$, $\beta \in B$) be a double numerical net. Suppose that for each fixed $\alpha \in A$ there exists the limit $\lim_{\beta} x_{\alpha\beta} = x_\alpha$, and for each fixed $\beta \in B$ there exists the limit $\lim_{\alpha} x_{\alpha\beta} = x_\beta$. Then, if for every $\varepsilon > 0$ there exists a pair of indices α_0, β_0 such that for all $\alpha \geq \alpha_0$, $\beta \geq \beta_0$

$$|x_{\alpha\beta} - x_\alpha| < \varepsilon,$$

then from the existence of one iterated limit there follows the existence of the other and their equality

$$\lim_{\alpha} x_{\alpha} = \lim_{\beta} x_{\beta}.$$

By the **positive cone** $\hat{L}_+$ in the completion $\hat{L}$ of the space $(L(X), \tau_H)$ we shall mean the set

$$\{\hat{u} \in \hat{L} : \langle \hat{u}, f \rangle \geq 0 \text{ for } f \geq 0\}.$$

We shall say that the space (X, ξ) satisfies **condition (M)** relative to a given cone D^+ of signed-positive measures, defined on some σ -algebra Σ of subsets of X , if the cone $\hat{L}_+$ is linearly isomorphic to the cone D^+ in such a way that, under this isomorphism, an element $\hat{u} \in \hat{L}_+$ and the corresponding measure $\mu \in D^+$ are related by

$$\langle \hat{u}, f \rangle = \int_X f d\mu$$

for every function $f \in P(X)$. We shall say that the space (X, ξ) satisfies **condition (MR)** if in condition (M) as D^+ one takes the cone of all signed-positive bounded regular ⁽⁴⁾ Borel measures.

Theorem 1. Every separated uniformly locally bicomact space (X, ξ) satisfies condition (MR).

Theorem 2. Every complete metrizable space (X, ξ) satisfies condition (MR).

We shall give the proof of the central part of this theorem, namely of the fact that every element $\hat{u}$ of the completion of the cone of positive elements in the space $(L(X), \tau_{\mathcal{H}})$ is a Daniel integral on the lattice $P(X)$.

Let $\{u_\alpha\}_{\alpha \in A}$ be a Cauchy net of positive elements of $(L(X), \tau_{\mathcal{H}})$, converging to $\hat{u}$. Consider a fundamental sequence of symmetric neighborhoods $V_1, V_2, \dots$ in ξ such that, for every n , $V_{n+1} \subset V_n$, and the sequence $\varepsilon/2^n$ ($n = 1, 2, \dots$), where $\varepsilon > 0$ is fixed.

Using Lemma 1, choose a sequence of indices α_n ($n = 1, 2, \dots$; $\alpha_n < \alpha_{n+1}$) such that

$$u_\beta[X \setminus \bigcup_n (Nu_{\alpha_n})] < \varepsilon/2^n$$

for all $\beta \geq \alpha_n$. Consider the sets

$$T_n = \bigcap_{k=1}^n V_k \left(\bigcup_{j=1}^n Nu_{\alpha_j} \right), \quad \Pi_n = \bigcap_{k=1}^n \times V_k^2 \left(\bigcup_{j=1}^k Nu_{\alpha_j} \right)$$

and

$$\Pi = \bigcap_{k=1}^{\infty} V_k^2 \left(\bigcup_{j=1}^k Nu_{\alpha_j} \right).$$

The set Π is totally bounded, and its closure $\overline{\Pi}$ is bicompat by virtue of the completeness of the space (X, ξ) .

By definition we have

$$\Pi_n = \Pi_{n-1} \cap V_n^2 \left(\bigcup_{j=1}^n Nu_{\alpha_j} \right), \quad T_n = T_{n-1} \cap V_n \left(\bigcup_{j=1}^n Nu_{\alpha_j} \right),$$

and therefore

$$T_n \subset V_n \left(\bigcup_{j=1}^n Nu_{\alpha_j} \cap V_n(T_{n-1}) \right).$$

Further,

$$V_n(T_{n-1}) = V_n \left(\bigcap_{k=1}^{n-1} V_k \left(\bigcup_{j=1}^k Nu_{\alpha_j} \right) \right) \subset \bigcap_{k=1}^{n-1} V_n \circ V_k \left(\bigcup_{j=1}^k Nu_{\alpha_j} \right) \subset \bigcap_{k=1}^{n-1} V_k^2 \left(\bigcup_{j=1}^k Nu_{\alpha_j} \right) = \Pi_{n-1}.$$

Thus

$$T_n \subset V_n \left(\bigcup_{j=1}^n Nu_{\alpha_j} \cap \Pi_{n-1} \right).$$

In turn,

$$\bigcup_{j=1}^n Nu_{\alpha_j} \cap \Pi_{n-1} \subset \bigcup_{j=1}^{n+1} Nu_{\alpha_j} \cap \Pi_n \subset \dots \subset \bigcup_{j=1}^m Nu_{\alpha_j} \cap \Pi_m \subset \dots,$$

so that

$$\bigcup_{j=1}^m Nu_{\alpha_j} \cap \Pi_{n-1} \subset \bigcap_{m=1}^{\infty} \Pi_m = \Pi$$

and $T_n \subset V_n(\Pi)$. For $\beta \geq \alpha_n$,

$$u_\beta[X \setminus T_n] = u_\beta \left[X \setminus \bigcap_{k=1}^n V_k \left(\bigcup_{j=1}^k Nu_{\alpha_j} \right) \right] = u_\beta \left[\bigcup_{k=1}^n X \setminus V_k \left(\bigcup_{j=1}^k Nu_{\alpha_j} \right) \right] \leq \sum_{k=1}^n \frac{\varepsilon}{2^k} < \varepsilon.$$

Now let $f_k(x)$ ($k = 1, 2, \dots$) be an arbitrary sequence of functions from $P(X)$ such that $f_k \downarrow 0$ ($\sup f_1(x) = c < \infty$). We have the double net $\langle u_\alpha, f_k \rangle$ ($\alpha \in A$; $k = 1, 2, \dots$). For each fixed $\alpha \in A$,

$$\lim_{k \rightarrow \infty} \langle u_\alpha, f_k \rangle = 0,$$

and for fixed k ,

$$\lim \langle u_\alpha, f_k \rangle = \langle \hat{u}, f_k \rangle.$$

Moreover, the set of sums

$$\sum_{x \in X} \lambda_x^\alpha = \langle u_\alpha, 1 \rangle \quad (\alpha \in A)$$

may be assumed bounded (by some $d < \infty$) in view of the convergence of the net $\langle u_\alpha, 1 \rangle$ ($\alpha \in A$).

The monotonically decreasing sequence $f_k(x)$ ($k = 1, 2, \dots$) converges uniformly on the bicomact set $\bar{\Pi}$. Choose, for the given $\varepsilon > 0$, a number k_0 such that

$$\sup_{x \in \bar{\Pi}} f_k(x) < \varepsilon$$

for all $k \geq k_0$, and a number n such that

$$|f_{k_0}(x) - f_{k_0}(y)| < \varepsilon$$

whenever $(x, y) \in V_n$. Hence, if $k > k_0$ and $x \in V_n(\bar{\Pi})$, then

$$f_k(x) \leq f_{k_0}(x) < 2\varepsilon.$$

In particular, this is true for all $x \in T_n$, since, as shown earlier, $T_n \subset V_n(\bar{\Pi})$. Thus, for every element

$$u_\alpha = \sum_{x \in X} \lambda_x^\alpha x \quad (\alpha \geq \alpha_n)$$

and for every function f_k ($k \geq k_0$),

$$\begin{aligned} \langle u_\alpha, f_k \rangle &= \sum_{x \in X} \lambda_x^\alpha f_k(x) = \sum_{x \in T_n} \lambda_x^\alpha f_k(x) + \sum_{x \notin T_n} \lambda_x^\alpha f_k(x) \leq \\ &\leq 2\varepsilon \sum_{x \in T_n} \lambda_x^\alpha + c \sum_{x \notin T_n} \lambda_x^\alpha < 2\varepsilon d + c\varepsilon = \varepsilon(c + 2d). \end{aligned}$$

This means that Lemma 2 is applicable to the double net $\langle u_\alpha, f_k \rangle$, and that

$$\lim_{k \rightarrow \infty} \langle u, f_k \rangle = \lim_{\alpha} \lim_{k \rightarrow \infty} \langle u_\alpha, f_k \rangle = 0,$$

as was required to prove.

Let (X, ξ) be the Tikhonov product of uniform spaces

$$\prod_{\alpha \in A} (X_\alpha, \xi_\alpha).$$

A **Borel cylinder** in (X, ξ) is any set B of the form

$$\prod_{\alpha \in A} B_\alpha,$$

where, for each α , B_α is a Borel set in (X_α, ξ_α) , and only for a finite number of indices α is B_α different from the whole space X_α . By the **Kolmogorov σ -algebra** Σ_k on (X, ξ) we shall mean the smallest σ -algebra containing all Borel cylinders.

Let $\alpha \in A$ be a fixed index. Every measure μ defined on Σ_k then determines a certain measure μ_α on the σ -algebra of all Borel sets in (X, ξ) . Namely, for every Borel set B_α in (X_α, ξ_α) ,

$$\mu_\alpha(B_\alpha) = \mu \left(B_\alpha \times \prod_{\beta \in A \setminus \alpha} X_\beta \right).$$

A sign-positive measure defined on Σ_k will be called **Kolmogorov** if, for every $\alpha \in A$, the corresponding measure μ_α is regular ⁽⁴⁾.

Theorem 3. *If each of the factors (X_α, ξ_α) satisfies condition (MR), then the product (X, ξ) satisfies condition (M) with respect to the cone K^+ of all Kolmogorov measures on Σ_k .*

Remark. From Theorems 1, 2, and 3 it follows, in particular, that if (X, ξ) is a product of locally bicomact spaces or of complete metrizable spaces, then every element $\hat{u}$ of the completion of the cone of positive elements in the space $(L(X), \tau_{\mathcal{H}})$ is a Daniell integral on the lattice $P(X)$. This result was obtained independently by V. P. Fedorova.

Let X be a completely regular topological space; let ξ_X be the strongest uniformity on X compatible with the given topology. Then the space $P(X)$ of all bounded uniformly continuous functions on (X, ξ_X) coincides with the space $\tilde{C}(X)$ of all bounded continuous functions on X , and the family $\mathcal{H}$ coincides with the family of all uniformly bounded and equicontinuous at each point subsets of $\tilde{C}(X)$.

Theorem 4. *If a completely regular space X is such that the corresponding uniform space (X, ξ_X) is complete, then every element $\hat{u}$ of the completion of the cone of positive elements in the space $(L(X), \tau_{\mathcal{H}})$ is a Daniell integral on the lattice $\tilde{C}(X)$.*

Example 1. If the space X is complete in some uniformity compatible with the given topology, then it is complete also in the uniformity ξ_X .

Example 2. If the space X is paracompact, then it is complete in the uniformity ξ_X .

Moscow State Pedagogical Institute
named after V. I. Lenin

Received
1 IV 1966

REFERENCES

¹ M. Katětov, Proc. Intern. Congr. Math., 1962. ² D. A. Raikov, Matem. sborn., **63**, No. 4, 582 (1964). ³ V. Pták, Czechoslovak Math. J., **14** (89), 562 (1964). ⁴ I. A. Berezanskii, DAN, **171**, No. 2 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.