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Abstract

Full Text

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PHYSICS

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EQUATIONS FOR GREEN' S FUNCTIONS IN COULOMB SYSTEMS WITH COLLISIONS

(Presented by Academician N. N. Bogolyubov on 24 II 1966)

An electrically neutral system with Coulomb interaction is considered, in which one species of particles moves in a cloud of spatially compensating charge of the other species. We shall start from the kinetic equation obtained from Bogolyubov' s general chain ⁽¹⁾ by Lenard ⁽²⁾, with the addition to it of a spatially inhomogeneous term

$$\frac{\partial F_1(t, x_1)}{\partial t} = [p_1^2/2m, F_1(t, x_1)] + [U(t, r_1), F_1(t, x_1)] + \frac{\partial}{\partial p_{1i}} \int dp_2 Q_{ij}(p_1, p_2) \left\{ \frac{\partial F_1(t, x_1)}{\partial p_{1j}} F_1(t, x_2) - \frac{\partial F_1(t, x_2)}{\partial p_{2j}} \right\}$$

where

$$U(t, r) = n \int \Phi(|r - r_1|) \{ \rho(t, r_1) - 1 \} dr_1; \quad \rho(t, r) = \int F_1(t, x) dp;$$

$x_i = \{r_i; p_i\}$; $Q_{ij}(p_1, p_2)$ is obtained from the Lenard kernel $Q_{ij}(v, v')$ by transforming the equation from the variables v, v' to p_1, p_2 and changing the normalization of the distribution function from n to 1. We note that the last term in (1) is obtained under the assumption that $F_2(t, x_1, x_2) = F_2(x_1, x_2 | F_1)$, i.e., for $t > \tau_{\text{syn}}$, where τ_{syn} is the synchronization time of the higher distribution functions with F_1 .

It is easy to show that the Maxwell distribution is a stationary solution of (1), not only in the spatially homogeneous case (see ⁽²⁾), but also in the inhomogeneous case. Suppose that in an equilibrium system an external field with Hamiltonian

$$H = \frac{n}{2\pi} e^{-iEt} \delta A(x) = \frac{n}{2\pi} \delta A(t, x)$$

is switched on adiabatically. Then, taking into account the presence of $\delta(kp_1 - kp_2)$ in Q_{ij} and in δQ_{ij} , for small variations $\delta F_1(t, x) = F_1(t, x) - F_1^0(p)$ we obtain from (1)

$$\frac{\partial \delta F_1}{\partial t} = \left[\frac{p^2}{2m}, \delta F_1 \right] + n \int [\Phi(|r-r'|) \delta F_1(t, x'); F_1^0(p)] dx' + \frac{n}{2\pi} [\delta A(t, x); F_1^0(p)] + \frac{\partial}{\partial p_i} \int dp_2 Q_{ij}^0(p, p_2) \times \left\{ \frac{\partial F_1}{\partial t} \right\}$$

where Q_{ij}^0 means that $F^0(p)$ has been substituted into it in place of $F_1(t, x)$. Using the properties of integrals of generalized functions, Q_{ij}^0 can be computed exactly in the coordinate system in which $(p_1 - p_2) \parallel k_z$. As a result we obtain

$$Q_{ij}^0(p_1, p_2) = \frac{e^4 m n}{|p_1 - p_2|} \int_0^{2\pi} d\varphi \chi_i \chi_j \frac{\text{Im}\{\chi^0 \ln[1 + (4\pi n r_d^3)^2 / \chi^0]\}}{\text{Im} \chi^0}, \quad (3)$$

where

$$\text{Re} \chi^0(u) = 1 - 2 \frac{u}{p_T^2} e^{-(u/p_T)^2} \int_0^u e^{(t/p_T)^2} dt, \quad \text{Im} \chi^0 = -\sqrt{\pi} \left(\frac{u}{p_T} \right) e^{-(u/p_T)^2}.$$

and the notation has been introduced: $\vec{\chi}$ is a unit vector along $\mathbf{k}$; $u = \chi p$; φ is the angle determining the position of $\mathbf{k}$ in the plane perpendicular to $(\mathbf{p}_1 - \mathbf{p}_2)$; r_d is the Debye radius. Since Q_{ij}^0 enters the integral only as the product $Q_{ij}^0 F_1^0(p/p_T)$, one may restrict oneself to its values at small u/p_T , i.e., take $\text{Re} \chi^0 = 1$ and $\text{Im} \chi^0 = 0$. Then, taking into account the inequality, explicit for Coulomb systems, $n r_d^3 \gg 1$, (3) becomes

$$Q_{ij}^0 = \frac{2\pi m n e^4}{|\mathbf{p}_1 - \mathbf{p}_2|} \ln(4\pi n r_d^3) \quad (ij = 11, 22).$$

In an arbitrary coordinate system,

$$Q_{ij}^0 = 2\pi m n e^4 g^{-3} (g^2 \delta_{ij} - g_i g_j),$$

where $\mathbf{g} = \mathbf{p}_1 - \mathbf{p}_2$. Thus, incidentally, we have obtained that the kernel of the linearized Bogolyubov–Lenard equation (1) becomes the kernel of the linearized Landau equation for $p \ll p_T$, although the equations themselves differ only slightly owing to the cutoff factor $F_1^0(p/p_T)$ for all p .

We next carry out the transition to equations for Green's functions with the aid of the theorem on the variation of the mean value of a dynamical quantity ⁽³⁾. As shown in ⁽⁴⁾, this theorem is also valid for classical systems. According to this theorem,

$$\delta F_1(t, x) / \delta A(t, y) = \langle \langle A_x | A_y \rangle \rangle_E, \quad \delta A(t, x) / \delta A(t, y) = \delta(x - y). \quad (4)$$

Varying (2) with respect to the external field and substituting (4), we obtain

$$\begin{aligned}
 -iE \langle \langle A_{x_1} | A_y \rangle \rangle_E &= [p_1^2/2m; \langle \langle A_{x_1} | A_y \rangle \rangle_E] + \\
 &+ n \frac{\partial F_1^0(p_1)}{\partial p_{1\alpha}} \int \frac{\partial \Phi(|\mathbf{r}_1 - \mathbf{r}_2|)}{\partial r_{1\alpha}} \left(\int \langle \langle A_{x_2} | A_y \rangle \rangle_E dp_2 \right) dr_2 + \frac{n}{2\pi} \frac{\partial \delta(x-y)}{\partial r_{1\alpha}} \frac{\partial F_1^0(p_1)}{\partial p_{1\alpha}} + \\
 &+ \frac{\partial}{\partial p_{1\alpha}} \int dp_2 Q_{\alpha\beta}^0(\mathbf{p}_1, \mathbf{p}_2) \left\{ \frac{\partial F_1^0(p_1)}{\partial p_{1\beta}} \langle \langle A_{x_2} | A_y \rangle \rangle_E + \frac{\partial \langle \langle A_{x_1} | A_y \rangle \rangle_E}{\partial p_{1\beta}} F_1^0(p_2) \right. \\
 &\quad \left. - \frac{\partial \langle \langle A_{x_2} | A_y \rangle \rangle_E}{\partial p_{2\beta}} F_1^0(p_1) - \frac{\partial F_1^0(p_2)}{\partial p_{2\beta}} \langle \langle A_{x_1} | A_y \rangle \rangle_E \right\}. \quad (5)
 \end{aligned}$$

The last equation is transformed by the Fourier method, i.e.,

$$\langle \langle A_{x_1} | A_y \rangle \rangle_E = \frac{1}{(2\pi)^3} \int G(k) e^{i\mathbf{k}\mathbf{r}} d\mathbf{k},$$

so that for the Fourier transforms in space we have

$$\begin{aligned}
 -iE G(E, k, p_1, p') &= -i \left(\frac{\mathbf{k}\mathbf{p}_1}{m} \right) G(E, k, p_1, p') + \\
 &+ ik_\alpha n \frac{\partial F_1^0(p_1)}{\partial p_{1\alpha}} \Phi(k) \int G(E, k, p_2, p') dp_2 + \quad (6) \\
 &+ \frac{\partial}{\partial p_{1\alpha}} \int dp_2 Q_{\alpha\beta}^0(\mathbf{p}_1, \mathbf{p}_2) \left\{ \frac{\partial F_1^0(p_1)}{\partial p_{1\beta}} G(E, k, p_2, p') + \frac{\partial G(E, k, p_1, p')}{\partial p_{1\beta}} F_1^0(p_2) \right. \\
 &\quad \left. - \frac{\partial G(E, k, p_2, p')}{\partial p_{2\beta}} F_1^0(p_1) - \frac{\partial F_1^0(p_2)}{\partial p_{2\beta}} G(E, k, p_1, p') \right\} + i \frac{k_\alpha n}{2\pi} \delta(p_1 - p') \frac{\partial F_1^0(p_1)}{\partial p_{1\alpha}}.
 \end{aligned}$$

Introducing

$$\hat{M} = \frac{\partial}{\partial p_\alpha} \left\{ \int dp_2 Q_{\alpha\beta}^0(p, p_2) \left[\frac{\partial F^0(p_2)}{\partial p_{2\beta}} - \frac{\partial}{\partial p_\beta} F^0(p_2) \right] \right\},$$

$$\mathcal{L}(E, k, p, p') = nk \frac{\partial F^0(p)}{\partial p} \Phi(k) \int G(E, k, p, p') dp + \frac{n}{2\pi} \delta(p - p') \mathbf{k} \frac{\partial F^0(p)}{\partial p},$$

$$L(E, k, p, p') = -i \frac{\partial}{\partial p_\alpha} \int dp_2 Q_{\alpha\beta}^0(p, p_2) \left\{ \frac{\partial G(E, k, p_2, p')}{\partial p_{2\beta}} F^0(p) - \frac{\partial F^0(p)}{\partial p_\beta} G(E, k, p_2, p') \right\}. \quad (7)$$

it can be brought to the form

$$\left(E - \frac{\mathbf{k}\mathbf{p}}{m} + i\hat{M}\right) G(E, k, p, p') + \mathcal{L}(E, k, p, p') + L(E, k, p, p') = 0. \quad (8)$$

The solution of (8) is symbolically written in the form

$$G(E, k, p, p') = -\frac{\mathcal{L} + L}{(E - \mathbf{k}\mathbf{p}/m + i\hat{M})}. \quad (9)$$

The initial equation (1) that we used was obtained to within the small parameter $\varepsilon = 4\pi e^2/\theta r_d$, where $r_d^{-2} = 4\pi e^2 n/\theta$.

Let us estimate the orders of magnitude of $\mathcal{L}$ and L in ε . To do this, in equation (8) we pass to dimensionless quantities, introducing $\tilde{p} = p/p_T$, where $p_T = \sqrt{2m\theta}$, $\tilde{k} = k/k_d$, $k_d^2 = r_d^{-2}$. Then $\mathcal{L} \sim e^2 n/p_T k_d$, $Q_{ij}^0 \sim mne^4 p_T^{-1} \ln(4\pi n r_d^3)$, and, consequently, $L \sim mne^4 \ln(4\pi n r_d^3)/p_T^3$,

$$L/\mathcal{L} \sim me^2 k_d \ln(4\pi n r_d^3)/p_T^2 = e^2 \ln(4\pi n r_d^3)/\theta r_d \sim \varepsilon \ln(4\pi n r_d^3).$$

Since $\ln(4\pi n r_d^3)$ is of the order of several units, while ε is small, in (9) L may be neglected in comparison with $\mathcal{L}$. Then

$$G(E, k, p, p') = -\frac{\mathcal{L}}{(E - \mathbf{k}\mathbf{p}/m + i\hat{M})}. \quad (10)$$

In other words, one may say that in equation (8) the first two terms have order G/t , and the last G/τ_{rel} , where $\tau_{\text{rel}} \sim nr_d^3/\omega_0 \ln(4\pi n r_d^3)$. Therefore expression (10) is valid for times $t < \tau_{\text{rel}}$. Together with Bogoliubov's condition of synchronization of correlation functions, we arrive at the conclusion that G in the form (10) is valid for $\tau_{\text{syn}} < t < \tau_{\text{rel}}$.

As is seen from (6) and (10), for $\varepsilon = 0$ we obtain for the Green functions the Vlasov equation with a self-consistent field and its solution, found by N. N. Bogoliubov, Jr., and B. Sadoynikov in ⁽⁴⁾. Taking the collision term into account, owing to the presence of $i\hat{M}$ in the denominator of (10), will give additional damping. Since $\hat{M} \sim \varepsilon$, we seek the solution (10) in the form $G = G^0 + \varepsilon G^1$, where G^0 is the solution of an equation of Vlasov type ⁽⁴⁾. For G^1 we obtain the equation

$$(E - \mathbf{k}\mathbf{p}/m)G^1(E, k, p, p') + \mathcal{L} + i\hat{M}G^0(E, k, p, p') = 0. \quad (11)$$

Integrating it with respect to p' ,

$$(E - \mathbf{k}\mathbf{p}/m)G^1(E, k, p) + \mathcal{L}(E, k, p) + i\hat{M}G^0(E, k, p) = 0; \quad (12)$$

$$G^0(E, k, p) = z(E, k) \frac{(\mathbf{k}\mathbf{p})e^{-(p/p_T)^2}}{(E - \mathbf{k}\mathbf{p}/m)}, \quad z = \frac{nc}{2\pi m\theta} \frac{1}{\left\{1 - \left(2knv(k)/\sqrt{2\pi m\theta}\right) B(E, k)\right\}} \quad (13)$$

Introducing $f^1(E, k) = \int G^1(E, k, p) dp$, after division of (12) by $(E - \mathbf{k}\mathbf{p}/m)$ and subsequent integration over p , we find that

$$f^1(E, k) = -i \int \frac{\hat{M}G^0 dp}{(E - \mathbf{k}\mathbf{p}/m)} \Big/ \left\{1 - \frac{2knv(k)}{\sqrt{2\pi m\theta}} B(E, k)\right\}. \quad (14)$$

Taking $\hat{M}$ from (7), after rather cumbersome calculations we obtain

$$\alpha = \int \frac{\hat{M}G^0 dp}{(E - \mathbf{k}\mathbf{p}/m)} = \frac{8\pi DzEk^2}{3m} \int_0^\infty \frac{\Phi(p/p_T) p e^{-(p/p_T)^2}}{(k^2 p^2/m^2 - E^2)^2} dp - \frac{16\pi DzEk^4 p_T^2}{3m^3} \int_0^\infty \frac{\Phi(p/p_T) p e^{-(p/p_T)^2}}{(k^2 p^2/m^2 - E^2)^3} dp + \frac{32\pi DzEk^6 p_T^4}{3m^4} \int_0^\infty \frac{\Phi(p/p_T) p e^{-(p/p_T)^2}}{(k^2 p^2/m^2 - E^2)^4} dp$$

$$\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad D = 2\pi m n e^4 \ln(4\pi n r_d^3).$$

For large k the integrals in (15) diverge. In the approximation of small k , and also $p \ll p_T$, since under this condition the replacement of (3) by the Landau kernel is valid, $a = 4\pi Dz k^2 p_T^2 / 3mE^3$,

$$f^1(E, k) = -i \frac{4n^2 k^2 e^4 \ln(4\pi n r_d^3)}{3\sqrt{2\pi} (m\theta)^{3/2} E^3 \mathcal{E}^2}, \quad (16)$$

where $\mathcal{E}$ is the denominator in (14). $f^0(E, k)$ (see (4)) at small k gives

$$f^0(E, k) = \frac{nk}{\pi\sqrt{2\pi m\theta}} \frac{B(E, k)}{\mathcal{E}} \simeq \frac{nk^2}{2\pi m E^2 \mathcal{E}}.$$

For $f^1(E, k)$ we shall finally have

$$f^1(E, k) = -i f^0(E, k) \frac{\nu_{\text{eff}}}{E\mathcal{E}}, \quad \nu_{\text{eff}} = \frac{4}{3} \frac{\sqrt{2\pi n e^4}}{m^{1/2} \theta^{3/2}} \ln(4\pi n r_d^3). \quad (17)$$

We note that ν_{eff} coincides with the result (5) for the case of frequencies smaller than ω_0 , where our consideration is also valid. The complete Green's function will be

$$f(E, k) = f^0 + f^1 = f^0 \left(1 - i \frac{\nu_{\text{eff}}}{E\mathcal{E}} \right) = \frac{f^0(E, k)}{(1 + i\nu_{\text{eff}}/E\mathcal{E})}. \quad (18)$$

It is known that the poles of the Green's function give the frequencies and damping of the system. Writing $E = \omega + i\gamma$, we find that for plasma waves at small k

$$\omega^2 = \omega_0^2 - \nu_{\text{eff}}^2/4 = \omega_0^2 \{ 1 - \varepsilon^2 \ln^2(4\pi n r_d^3)/288\pi^3 \}; \quad (19)$$

$$\gamma = -\gamma_L + \gamma_{\text{coll}} = -\gamma_L - \nu_{\text{eff}}/2 = -\gamma_L - \sqrt{2\pi\varepsilon} \omega_0 \ln(4\pi n r_d^3)/24\pi^2, \quad (20)$$

where γ_L is Landau damping. As was to be expected, $\gamma_{\text{coll}} \sim \varepsilon$, i.e. the inequality $\gamma_{\text{coll}} \ll \omega$ is satisfied.

The next approximation in k gives a correction to the Vlasov frequency and damping with dependence on the wave vector, along with Landau damping,

$$\gamma' = -\frac{\sqrt{2\pi\varepsilon} \omega_0 \ln(4\pi n r_d^3)}{24\pi^2} \left(1 + \frac{3\theta k^2}{m\omega_0^2} \right) - \gamma_L, \quad (21)$$

$$\omega'^2 = \omega_0^2 \left[1 - \frac{\varepsilon^2 \ln^2(4\pi n r_d^3)}{288\pi^3} \right] + \frac{3\theta k^2}{m} \left[1 - \frac{\varepsilon^2 \ln^2(4\pi n r_d^3)}{144\pi^3} \right]. \quad (22)$$

These calculations were carried out for a Landau-type kernel. If one takes the more accurate approximation (3), which correctly conveys the general behavior and, in particular, preserves the asymptotics of the integrand at small and large momenta, then ν_{eff} will be smaller by a factor of $\sqrt{2}$, and instead of (21) and (22) we obtain

$$\gamma'' = -\frac{\sqrt{\pi\varepsilon} \omega_0 \ln(4\pi r_d^3)}{24\pi^2} \left(1 + \frac{4\theta k^2}{m\omega_0^2} \right) - \gamma_L,$$

$$\omega''^2 = \omega_0^2 \left[1 - \frac{\varepsilon^2 \ln^2(4\pi r_d^3)}{576\pi^3} \right] + \frac{3\theta k^2}{m} \left[1 - \frac{\varepsilon^2 \ln^2(4\pi r_d^3)}{216\pi^3} \right].$$

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REFERENCES

1. N. N. Bogolyubov, *Problems of Dynamical Theory in Statistical Physics*, 1947.
2. A. Lenard, *Ann. Phys.*, **10**, 3 (1960).
3. D. N. Zubarev, *Sov. Phys. Usp.*, **71**, 1 (1960).
4. N. N. Bogolyubov, Jr., B. I. Sadovnikov, *ZhETF*, **43**, 8 (1962).
5. V. P. Silin, *ZhETF*, **41**, 3 (9) (1961).

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