

# ON A PROBLEM OF MIXED TYPE FOR THE EQUATION

$$y(y-1)u_{xx} + u_{yy} = 0$$

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## Abstract

## Full Text

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MATHEMATICS

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# ON A PROBLEM OF MIXED TYPE FOR THE EQUATION $y(y-1)u_{xx} + u_{yy} = 0$

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The equation

$$L(u) \equiv k(y)u_{xx} + u_{yy} = 0, \quad (L)$$

where  $k(y) = y^2 - y$ , is an equation of mixed type; it is elliptic for  $y < 0$  and  $y > 1$ , hyperbolic for  $0 < y < 1$ , and along the straight lines  $y = 0$ ,  $y = 1$  it degenerates parabolically. The families of curves  $x \pm \frac{1}{4}k'\sqrt{-k} \pm \frac{1}{8}\arcsin k' = \text{const}$  are real characteristics of equation (L), while the straight lines  $y = 0$ ,  $y = 1$  constitute the locus of points of return of this family.

1. We introduce the following notation:

$$\omega(y) = \frac{1}{4}k'\sqrt{-k} + \frac{1}{8}\arcsin k', \quad \tilde{y}(y) = -(3/2a)^{2/3}(\omega \mp \pi/16)^{2/3},$$

$$0 \leq y \leq 1, \quad (1)$$

$$\omega(y) = \frac{1}{4}k'\sqrt{k} - \frac{1}{8}\ln |k' + 2\sqrt{k}|, \quad \tilde{y}(y) = (3\omega/2a)^{2/3},$$

$$y \leq 0, \quad y \geq 1;$$

$$\xi(x, y) = (x + \omega - \pi/16)/a, \quad \eta(x, y) = (x - \omega + \pi/16)/a,$$

$$a = \text{const}, \quad \tilde{x} = x/a; \quad (2)$$

$$\pi/8 < a < \pi/4, \quad b(\tilde{y}) = [\tilde{y}'(y)]^{-2} \tilde{y}'''(y), \quad \tilde{\omega}(y) = \exp \left( \frac{1}{2} \int_0^{\tilde{y}} b(s) ds \right).$$

In the plane of the variables  $x, y$ , consider the points  $A_1(0, 0)$ ,  $A_0(\pi/8, 0)$ ,  $A_2(a, 0)$ ,  $B(a - \pi/16, 1/2)$ ,  $A_3(a, 1)$ ,  $A(a - \pi/8, 1)$ ,  $A_4(0, 1)$ ,  $B_1(\pi/16, 1/2)$ . Let  $D^*$  be a bounded simply connected domain bounded by: 1) the arc  $\sigma_0$ :  $(x - a/2)^2 + \omega^2 = a^2/4$ ,  $y \leq 0$ ; 2) the arc  $\sigma_1$ :  $(x - a/2)^2 + \omega^2 = a^2/4$ ,  $y \geq 1$ ; 3) the characteristics  $A_1B_1$ :  $\eta = \pi/8a$ ,  $B_1A_4$ :  $\xi = 0$ ,  $A_2B$ :  $\xi = 1 - \pi/8a$ ,  $BA_3$ :  $\eta = 1$  of equation (L). Let, further,  $B_0(D)$  be the point of intersection of the characteristics  $A_0B_0$ :  $\eta = \pi/4a$  and  $AB_0$ :  $\xi = 1 - \pi/8a$  ( $AD$ :  $\eta = 1 - \pi/8a$  and  $B_1A_4$ );  $D$  be the subdomain of  $D^*$  lying above the curve  $DAB$ ;  $D^+$  the elliptic part of the domain  $D$ ;  $\Delta$  ( $\Delta^*$ ) the hyperbolic part of  $D$  lying above the curve  $A_4DA$  ( $ABA_3$ );  $D^-$  the domain of the characteristic quadrilateral  $AB_0A_0D$ .

Following A. V. Bitsadze <sup>(1)</sup>, by a solution of equation (L) that is regular in the domain  $D^*$  we shall mean a solution  $u(x, y)$  possessing the following properties: 1)  $u$  is continuous everywhere in the closed domain  $\overline{D^*}$ , except at the points  $A, A_0, A_2$  and, possibly, at the points  $A_1, A_3, A_4$ , where it tends to  $\infty$  of order  $\alpha < 1/6$ ; 2)  $u_y$  ( $u_x$ ) is continuous everywhere in  $\overline{D^*}$ , except, possibly, at the points  $A, A_i$ ,  $i = 0, \dots, 4$ , and at the characteristics issuing from these points, where it tends to  $\infty$  of order  $< 5/6$  ( $< 7/6$ ) and  $< 1$  ( $< 1$ ), respectively; 3)  $u$  is twice continuously differentiable everywhere in  $D^*$ , except, possibly, at the boundary of the domain  $D^-$ .

**Problem 1.** Find a regular solution in the domain  $D^*$  of equation (L), satisfying the conditions:

$$(\tilde{\omega}u)_{\sigma_0} = \varphi_0(\tilde{x}), \quad (\tilde{\omega}u)_{\sigma_1} = \varphi_1(\tilde{x}), \quad 0 < \tilde{x} < 1; \quad (3)$$

$$(\tilde{\omega}u)_{AB_0} = \psi_1(\eta), \quad x_0 < \eta \leq \pi/4a; \quad (\tilde{\omega}u)_{AD} = \psi_2(\xi),$$

$$0 \leq \xi < x_0 = 1 - \pi/8a,$$

where  $\tilde{\omega}$  is given by formulas (1), (2), with in (1) the minus sign taken before  $\pi/16$ , and  $\varphi_i$  and  $\psi_i$  are representable in the form

$$\varphi_0(\tilde{x}) = \alpha_0(\tilde{x})\tilde{x}^{-\chi_0}(1 - \tilde{x})^{-\alpha/2}, \quad \varphi_1(\tilde{x}) = \alpha_1(\tilde{x})(\tilde{x} - \tilde{x}^2)^{-\chi_1},$$

$$\psi_1(\eta) = \beta_1(\eta)(\eta - x_0)^{-\alpha}(\pi/4a - \eta)^{1/6-\alpha}, \quad \psi_2(\xi) = \beta_2(\xi)(x_0 - \xi)^{-\alpha},$$

where

$$\alpha_i \in C^{(2)} \quad (0 \leq \tilde{x} \leq 1), \quad \alpha_0(1) \neq 0; \quad \beta_1 \in C^{(4)} \quad (x_0 \leq \eta \leq \pi/4a),$$

$$\beta_i(x_0) \neq 0, \quad \beta_1(\pi/4a) \neq 0, \quad \beta_2 \in C^{(4)} \quad (0 \leq \xi \leq x_0), \quad 1/12 > \chi_i = \text{const.}$$

**2. Uniqueness of the solution of Problem 1** follows directly from the uniqueness of the solution of the Tricomi problem for equation (L) in the domain  $D^-$  with data on the characteristics  $AB_0, AD$ , and from the following two lemmas.

**Lemma 1.** Let  $u(x, y)$  be a regular solution in the domain  $\Delta$  ( $\Delta^*$ ) of equation (L), vanishing on  $AD$  ( $AB$ ). Then

$$\int_0^{x_1} u(x, 1) u_y(x, 1) dx \geq 0 \quad \left( \int_{x_1}^a u(x, 1) u_y(x, 1) dx \geq 0 \right), \quad x_1 = a - \pi/8.$$

**Lemma 2.** Let  $u(x, y)$  be a regular solution in the domain  $D^+$  of equation (L), vanishing on  $\sigma_1$ . Then

$$\int_0^a u(x, 1) u_y(x, 1) dx \leq 0.$$

The validity of Lemma 1 is established by the method of F. I. Frankl<sup>(2)</sup>. Lemma 2 is proved in the same way as in the case of the Tricomi equation<sup>(2)</sup>.

### 3. Proof of existence of the solution.

**Lemma 3.** Let: 1)  $u \in C(\overline{D^+})$ ; 2)  $u \in C^{(1)}(D^+ - A_3 - A_4)$ ; 3)  $u \in C^{(2)}(D^+)$ ,  $L(u) \equiv 0$  in  $D^+$ ; 4)  $(u)_{\sigma_1} = 0$ ,  $u_y(x, 1) + {}^1/5 u(x, 1) = 0$  for  $0 < x < a$ . Then  $u \equiv 0$  in  $\overline{D^+}$ .

**Lemma 4.** Let  $u$  satisfy the conditions of Lemma 1 and

$$u_y(x, 1) + {}^1/5 u(x, 1) = 0$$

for  $0 < x < x_1$ . Then  $u \equiv 0$  in  $\Delta$ .

Lemma 3 follows simply from the known Zaremba-Giraud extremum principle<sup>(3,4)</sup>, and Lemma 4 is a consequence of Lemma 1.

In equation (L) we pass to Frankl's variables<sup>(5)</sup>  $\tilde{x}, \tilde{y}$  and to the new function  $v = \tilde{\omega}u$ . Then  $v$  will satisfy the equation

$$\tilde{y} v_{\tilde{x}\tilde{x}} + v_{\tilde{y}\tilde{y}} + c(\tilde{y})v = 0, \quad (\tilde{L})$$

where

$$c(\tilde{y}) = -1/2 (b_{\tilde{y}} + 1/2 b^2),$$

and the domain  $D (D^+, \Delta, \Delta^*)$  passes into the domain  $\widetilde{D} (\widetilde{D}^+, \widetilde{\Delta}, \widetilde{\Delta}^*)$ , bounded by the normal contour

$$\tilde{\sigma}_1 : (\tilde{x} - 1/2)^2 + 4/9 \tilde{y}^3 = 1/4$$

and the characteristics  $\widetilde{A}_4 \widetilde{D}$ ,  $\widetilde{D} \widetilde{A}$ ,  $\widetilde{A} \widetilde{B}$ ,  $\widetilde{B} \widetilde{A}_3$  of equation  $(\widetilde{L})$ .

It is easy to show that

$$\tilde{y} = (y - 1) a^{-2/3} F^{2/3} (-1/2, 3/2, 5/2, 1 - y)$$

in  $\overline{D}$ ; consequently,  $c(\tilde{y})$  for

$$\tilde{y} > -(3\pi/16a)^{2/3} \quad (y > 0)$$

by virtue of (2) admits derivatives of any order, and  $c(0) > 0$ .

**Problem (1,1).** Find a regular solution in the domain  $\widetilde{D}^+$  of equation  $(\widetilde{L})$ , satisfying the conditions:

$$(v)_{\tilde{\sigma}_1} = \varphi_1(\tilde{x}) \quad \text{for } 0 < \tilde{x} < 1,$$

$$(v_{\tilde{y}})_{\tilde{y}=0} = \nu(\tilde{x}) \quad \text{for } 0 < \tilde{x} < x_0, \quad x_0 < \tilde{x} < 1.$$

**Problem (1,2).** Find a regular solution in the domain  $\widetilde{\Delta} (\widetilde{\Delta}^*)$  of equation  $(\widetilde{L})$ , satisfying the conditions

$$(v)_{\widetilde{A}\widetilde{D}} = \psi_2(\xi) \quad \text{for } 0 \leq \xi < x_0;$$

$$(v_{\tilde{y}})_{\tilde{y}=0} = \nu(\tilde{x}) \quad \text{for } 0 < \tilde{x} < x_0 \quad (v = \psi_1(\eta) \text{ on } \widetilde{A}\widetilde{B} \text{ for } x_0 < \eta \leq 1; v_{\tilde{y}}^2 = \nu(\tilde{x}) \text{ for } \tilde{y} = 0 \text{ and } x_0 < \tilde{x} < 1).$$

In both problems it is assumed that  $\nu(\tilde{x}) \in C^{(1)} (0 < \tilde{x} < x_0, x_0 < \tilde{x} < 1)$ , but may tend to infinity of order  $< 5/6$  for  $\tilde{x} = 0, x_0, 1$ .

The uniqueness of the solution of problems (1,1) and (1,2) is established with the aid of Lemmas 3 and 4, while the proof of existence is carried out according to the scheme proposed by Gellerstedt in [6].

**Lemma 5.** Suppose: 1) there exist solutions of Problem 1 in the domain  $D$ ; 2)  $\tau(x) = (v)_{\tilde{y}=0}$ ,  $\nu(x) = (v_{\tilde{y}})_{\tilde{y}=0}$ . Then

$$\tau(x) = \gamma \int_0^1 \nu(t) [L(x, t) - |t - x|^{-1/3} + (t + x - 2tx)^{-1/3}] dt + \Phi(x), \quad 0 < x < 1; \quad (4)$$

$$\tau(x) = \gamma \int_x^{x_0} \nu(t)(t - x)^{-1/3} P((t - x)^{4/3}) dt + \Psi_2(x), \quad 0 < x < x_0; \quad (5)$$

$$\tau(x) = \gamma \int_{x_0}^x \nu(t)(x - t)^{-1/3} P((x - t)^{4/3}) dt + \Psi_1(x), \quad x_0 < x < 1; \quad (6)$$

where: 1)  $\gamma \equiv \text{const}$ ,  $L \in C$  ( $0 \leq x, t \leq 1$ ),  $L \in C^{(2)}$  ( $0 < x, t < 1$ ); moreover for  $x = t$  ( $x = t = 0, 1$ ) the first derivatives may tend to infinity as  $\ln|x - t|$  ( $(t + x - 2tx)^{-1/3}$ ); 2)  $\Phi \in C^{(3)}$  ( $0 < x < 1$ ) and for  $x = 0, 1$  tends to infinity of order  $2\chi_1$ ; 3)  $P((x - t)^{4/3})$  admits derivatives of any order for  $x \neq t$ , moreover  $P(0) = 1$ , and the first derivatives for  $x = t$  tend to zero of order not less than  $1/3$ ; 4)  $\Psi_2 \in C^{(3)}$  ( $0 \leq x < x_0$ ),  $\Psi_1 \in C^{(3)}$  ( $x_0 < x \leq 1$ ), and for  $x = x_0$   $\Psi_i$  tends to infinity of order  $\alpha$ .

Lemma 5 follows directly from the constructive properties of the solution of problems (1,1) and (1,2).

From (4), (5), and (6), after a series of transformations entirely analogous to the transformations used in the case of the Tricomi problem, for determining  $\nu(x)$  we obtain the singular integral equation

$$\begin{aligned} \nu(x) = \lambda \int_0^{x_0} \nu(t) \left[ \left( \frac{t - x_0}{x - x_0} \right)^{2/3} \frac{1}{t - x} + \left( \frac{t + x_0 - 2tx_0}{x - x_0} \right)^{2/3} \frac{1}{t + x - 2tx} \right] dt - \lambda \int_{x_0}^1 \nu(t) \left[ \left( \frac{t - x_0}{x - x_0} \right)^{2/3} \frac{1}{t - x} \right. \\ \left. + \lambda \int_0^1 \nu(t) k(x, t) dt + h(x), \quad \lambda = \frac{1}{\pi\sqrt{3}}, \right. \end{aligned}$$

where  $h(x) \in C^{(2)}$  ( $0 < x < x_0$ ,  $x_0 < x < 1$ );  $h(x)$  for  $x = x_0$  tends to infinity of order  $\alpha + 2/3$  and to infinity of the same order at the points  $x = 0$ ,  $x = 1$ , if  $2\chi_1 = \alpha$ ; the kernel  $k(x, t)$  has the following properties: 1) it is regular for  $0 < x, t < 1$  and  $x \neq t$ ; for  $x = t$  the first derivatives may tend to infinity of order  $\leq \varepsilon + 2/3$ ; 2) for  $0 < x, t < 1$ ,  $x \neq x_0$ , it satisfies the inequality

$$|k(x, t)| \leq M_1(x - x_0)^{-2/3} [|L(x_0, t)| + |x - x_0|^{1-\varepsilon} |\ln(t + x - 2tx)|] + M_2,$$

where  $0 < M_i = \text{const}$ , and  $\varepsilon$  is an arbitrarily small positive number.

By the change of variable of integration  $t = (s + 1)/2$ , equation (7) can be rewritten in the form

$$\begin{aligned} \tilde{v}(y) = \lambda \int_{-1}^{y_0} \tilde{v}(s) \left[ \left( \frac{s - y_0}{y - y_0} \right)^{2/3} \frac{1}{s - y} + \left( \frac{1 - sy_0}{y - y_0} \right)^{2/3} \frac{1}{1 - sy} \right] ds - \\ - \lambda \int_{y_0}^1 \tilde{v}(s) \left[ \left( \frac{s - y_0}{y - y_0} \right)^{2/3} \frac{1}{s - y} - \left( \frac{1 - sy_0}{y - y_0} \right)^{2/3} \frac{1}{1 - sy} \right] ds + \lambda \int_{-1}^1 \tilde{v}(s) \tilde{k}(y, s) ds + \tilde{h}(y), \end{aligned} \quad (8)$$

where  $y = 2x - 1$ ,  $y_0 = 2x_0 - 1$ ,  $\tilde{v}(y) = v((y + 1)/2)$ .

Equation (8) for  $\tilde{k}(y, s) \equiv 0$  was studied by Gellerstedt (7). Using his results and relying on the uniqueness of the solution of problems 1; 1.1; 1.2, it is not difficult to show that equation (7) is unconditionally solvable (in the class of sought solutions) and that its solution  $v(x) \in C^{(1)}$  ( $0 < x < x_0$ ,  $x_0 < x < 1$ ), moreover it becomes infinite of order  $\alpha + 2/3$  for  $x = 0$ ,  $x_0$ , 1.

Knowing  $v(x)$  from (7), with the help of problems 1.1 and 1.2 we prove the existence of a solution of problem 1 in the domain  $D$ .

The construction of the solution  $u(x, y)$  of problem 1 in the domain  $D^-$  presents no essential difficulty, and therefore we restrict ourselves to giving the final result.

There exists a solution  $u(x, y)$  of problem 1 in the domain  $D^-$ , and it has the following properties: 1)  $u \in C(\bar{D}^- - A - A_0)$ , and at the points  $A$ ,  $A_0$  it becomes infinite of order  $\alpha$ ; 2)  $\partial u / \partial \xi$  is continuous everywhere in the closed domain  $\bar{D}^-$ , except for the characteristic  $AB_0$  and the point  $A_0$ , where it becomes infinite of order  $\alpha + 5/6$  for  $y \neq 1$  and infinite of order  $\alpha + 1$  at the points  $A$ ,  $A_0$ ; 3)  $\partial u / \partial \eta$  is continuous everywhere in  $\bar{D}^-$ , except for the characteristics  $AD$ ,  $A_0B_0$ , where it becomes infinite of order  $\alpha + 5/6$  for  $y \neq 1$ ,  $y \neq 0$  and infinite of order  $\alpha + 1$  at the points  $A$ ,  $A_0$ ; 4) let  $(\omega u)_{A_0B_1} = \tilde{\psi}_1(\eta)$ ,  $\pi/8a \leq \eta < \pi/4a$ ;  $(\omega u)_{A_0B_0} = \tilde{\psi}_2(\xi)$ ,  $0 < \xi \leq x_0$ , while  $\omega$  is given by formulas (1), (2), where before  $\pi/16$  the sign + remains. Then

$$\tilde{\psi}_1^{(i)}(\eta) = b_i(\eta)(\pi/4a - \eta)^{-\alpha-i}, \quad i = 0, 1, 2, 3, 4;$$

$$\tilde{\psi}_2^{(i)}(\xi) = a_i(\xi)\xi^{-\alpha-i}(x_0 - \xi)^{1/6-\alpha-i}, \quad i = 1, 2, 3, 4;$$

$$\tilde{\psi}_2^{(i)}(\xi) = a_0(\xi)\xi^{-\alpha}, \quad i = 0,$$

where

$$b_i \in C(\pi/8a \leq \eta \leq \pi/4a), \quad a_i \in C(0 \leq \xi \leq x_0), \quad b_i(\pi/4a) \neq 0,$$

$$a_i(0) \neq 0,$$

( $i$ ) denotes the order of the derivative.

The construction of the solution  $u(x, y)$  of problem 1 in the domain  $D^* - (D + D^-)$  is carried out in the same way as in the domain  $D$ , proceeding only from the enumerated properties of the function  $u$  on the characteristics  $A_0B_1$  and  $A_0B_0$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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