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Abstract

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MATHEMATICS

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BEHAVIOR OF A SPATIAL QUASICONFORMAL MAPPING ON PLANE SECTIONS OF THE DOMAIN OF DEFINITION

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In this note the concept of a plane quasiconformal mapping is generalized to the case of mappings of plane domains into 3-dimensional Euclidean space, and, with the aid of this concept, a characterization is given of the behavior of spatial quasiconformal mappings on plane sections of the domains of definition.

By **spatial quasiconformal mappings** we mean topological mappings of domains in 3-space which are quasiconformal in one of the known senses ⁽¹⁻³⁾; by a spatial K -quasiconformal mapping we mean a quasiconformal mapping whose principal linear part at almost all points of differentiability transforms the unit sphere into an ellipsoid with the ratio of the largest semiaxis to the smallest not exceeding the number K .

1°. **Definition 1.** Let, in 3-dimensional Euclidean space, a plane τ be given, and let in some domain G on this plane there be given a one-to-one and bicontinuous* mapping $y = f(x)$ into the space E_3 . We shall call $y = f(x)$ **2-dimensional K -quasiconformal** if there exists a rectangular coordinate system (u, v) on the plane τ such that: 1) $f(u, v)$ is differentiable almost everywhere in G ; 2) at almost all points of G the inequality $\max |\partial f / \partial l|^2 \leq KI$ holds, where

$$I = [(y'_{2u}y'_{3v} - y'_{2v}y'_{3u})^2 + (y'_{3u}y'_{1v} - y'_{3v}y'_{1u})^2 + (y'_{1u}y'_{2v} - y'_{1v}y'_{2u})^2]^{1/2};$$

- 3) $f(u, v)$ is an absolutely continuous function in the sense of Tonelli with square-summable partial derivatives in any rectangle with sides parallel to the coordinate axes whose closure belongs to G . The mapping $y = f(x)$ is 2-dimensional quasiconformal if it is K -quasiconformal for some K .

Using this definition, the class of spatial quasiconformal mappings can be characterized as follows:

Theorem 1. *In order that a topological mapping $y = f(x)$ of the ball $|x| < 1$ be spatial K -quasiconformal, it is necessary and sufficient that, for every

direction l , the mappings induced by the mapping l on almost all plane sections of the ball orthogonal to l be 2-dimensional K -quasiconformal.**

The set of directions occurring in Theorem 1 can be reduced.

Theorem 2. *Let $y = f(x)$ be a topological mapping of the ball $|x| < 1$, and let $\{l_n\}_{n=0}^\infty$ be a countable set of directions in 3-space possessing the following properties: 1) the directions l_n , $n = 1, 2, \dots$, belong to the set of all directions of some plane τ and form everywhere*

* Continuity here means continuity with respect to the set.

** For simplicity of formulation we consider here only quasiconformal mappings of the unit ball.

a subset dense in it; 2) the direction l_0 is orthogonal to the plane τ ; 3) the mappings induced by the mapping $f(x)$ on almost all plane sections of the ball orthogonal to l_n ($n = 0, 1, \dots$) are K -quasiconformal. Then $y = f(x)$ is a spatial K^2 -quasiconformal mapping.

2°. The question arises: can one describe the class of all spatial quasiconformal mappings on the basis of their behavior on plane sections of the domains of definition, orthogonal to directions from some finite set of directions in space? The answer is negative: for any finite set of directions there exists a counterexample. However, if one restricts oneself to the consideration of spatial K -quasiconformal mappings with K close to 1, then the following assertion can be proved.

Theorem 3. Let $\{y(x)\}$ be the set of all topological mappings of the ball $|x| < 1$. There exist a number K_0 , $2/\sqrt[3]{3} \leq K_0 \leq \sqrt[3]{3}$, and a finite single-valued real function $\varphi(K)$, defined on the interval $[1, K_0)$, having the following properties: 1) $\varphi(K) \geq 1$ for all K from $[1, K_0)$; 2) $\lim_{K \rightarrow 1} \varphi(K) = \varphi(1) = 1$; 3) if $1 \leq K < K_0$, then from the 2-dimensional K -quasiconformality of the mapping $y(x)$ from the set $\{y(x)\}$ on almost all plane sections of the ball orthogonal to the direction l_n ($n = 1, 2, 3$), where $\{l_n\}_{n=1}^3$ is some triple of mutually orthogonal directions in 3-space, it follows that this mapping is spatially $\varphi(K)$ -quasiconformal; 4) for $K > K_0$ there exists no number $\varphi(K)$ having property 3).

For conformal mappings in 3-space, from Theorem 3 we obtain the following characterization.

Theorem 4. In order that a topological mapping $y = f(x)$ of the ball $|x| < 1$ be conformal, it is necessary and sufficient that there exist in space such a triple of mutually orthogonal directions l_n , $n = 1, 2, 3$, that the mappings induced by the mapping under consideration on almost all plane sections of the ball orthogonal to l_n ($n = 1, 2, 3$) are 2-dimensional 1-quasiconformal mappings.

Simple examples show that in the conditions of Theorems 3 and 4 one cannot reduce the number of directions (with the property indicated in them) without violating the validity of these theorems.

3°. P. P. Belinskii constructed, for any $K > 1$, an example of a plane K -quasiconformal mapping of the disk $|z| < 1$ onto itself which carries some diameter of this disk into a curve that is not rectifiable in any of its parts ⁽⁴⁾.

We propose the following generalization of this example to the case of spatial quasiconformal mappings.

Theorem 5. For any number $K > 1$ there exists a K -quasiconformal mapping of the ball $R = \{|x| \leq 1\}$ onto itself, carrying its diametral section $\tau_3 = \{|x| < 1, x_3 = 0\}$ into a surface that is not quadrable in the sense of Lebesgue in any of its parts.

Proof. Fix a number $c, c > 1$, and for an arbitrary number ε ,

$$< \varepsilon \leq \frac{\sqrt{2} - 1 - \sqrt{1 + c^2} + c}{1 + (\sqrt{2} - 1)(\sqrt{1 + c^2} - c)},$$

consider a topological mapping $\Phi_\varepsilon(x)$ of the ball $|x| \leq 1$, which transforms each diametral section τ of the ball containing the axis x_3 onto itself, and on the section τ is represented as follows:

$$\Phi_\varepsilon(z) = \begin{cases} \frac{z + \varepsilon i}{\varepsilon i z + 1}, & \text{for } z \in \overline{D}_\tau, \\ z + \varepsilon i \frac{1 - z_{\partial D}^2}{\varepsilon i z_{\partial D} + 1} \frac{z_{\partial R} - z}{z_{\partial R} - z_{\partial D}}, & \text{for } z \in \overline{\tau}(\{|x| \leq 1\} - \overline{D}), \end{cases}$$

where for the points of the section τ the complex notation $z = u + iv$ is used (the axis v coincides with the axis x_3 , and the axis u is one of the two orthogonal axes x in the plane of the axes τ , passing through the point $(0, 0, 0)$), $z_{\partial D}$ is the point of inter-

section of a ray issuing from the origin and passing through the point z , with the boundary ∂D of the domain

$$D = \{x : x_1^2 + x_2^2 + (x_3 + c)^2 < 1 + c^2, x_1^2 + x_2^2 + (x_3 - c)^2 < 1 + c^2\};$$

$z_{\partial R}$ is the point of intersection of this ray with the surface ∂R of the ball R .

$\Phi_\varepsilon(x)$ leaves the points of the sphere $|x| = 1$ fixed, is conformal in the layer D , and is $q(\varepsilon)$ -quasiconformal in the whole ball, where $q(\varepsilon) \xrightarrow{\varepsilon \rightarrow 0} 1$.

Let now K be any number greater than 1. Fix the mapping $\Phi_\varepsilon(x)$ with $q(\varepsilon) \leq K$. We shall construct the desired mapping by means of superpositions of mappings of this kind: inside a ball of radius r it coincides (up to auxiliary rotations in the x - and y -spaces) with $r\Phi_\varepsilon(x/r)$, and outside it is the identity.

1st step. $\Phi_\varepsilon(x)$ increases the area of τ_3 and of spherical surfaces of sufficiently large radius, stretched over the circle $\{x_1^2 + x_2^2 = 1, x_3 = 0\}$, by a factor of $(1 + a_\varepsilon)$, $a_\varepsilon > 0$. Therefore, using superpositions of mappings of the indicated

kind, one can construct a K -quasiconformal mapping $f_1(x)$ of the ball R onto itself which is conformal in some neighborhood of each point of τ_3 , except for the points of a finite number of pairwise nonintersecting circles, and increases the area of τ_ε by no less than a factor of 2.

n -th step. $f_{n-1}(x)$ is conformal in a neighborhood of each point of τ_3 , except for the points of a finite set of pairwise nonintersecting circles. Therefore the set $f_{n-1}(\tau_3)$ consists of a finite number of spherical surfaces. Cover $f_{n-1}(\tau_3)$ by a finite number of (open) balls so that the following conditions are satisfied: 1) the closures of the balls have no common points with one another or with the lines of contact of the spherical pieces of the surface $f_{n-1}(\tau_3)$ and are contained in the domains of conformality of the mapping $f_{n-1}^{-1}(y)$; 2) the surface of each of these balls intersects the surface $f_{n-1}(\tau_3)$ in the circumference of a great circle; 3) the radii of the balls are so small that there exists a mapping of type $\Phi_\varepsilon(x)$, conformal in some neighborhood of the part of the surface $f_{n-1}(\tau_3)$ cut out by the balls, and increasing its area by a factor of $(1 + a_\varepsilon)$; 4) the area of the part of the surface $f_{n-1}(\tau_3)$ not covered by the balls, the area of its preimage, and the radii of the circles that are preimages of the sections of the balls with $f_{n-1}(\tau_3)$ do not exceed $1/n$. Applying these coverings and a finite number of superpositions of functions of type $\Phi_\varepsilon(x)$, we obtain a K -quasiconformal mapping $f_n(x)$, conformal in a neighborhood of each point of τ_3 , except for the points of a finite number of pairwise nonintersecting circles, and increasing the area of each of the spherical pieces of the surface $f_{n-1}(\tau_3)$ by no less than a factor of 2. In constructing $f_n(x)$ we shall also ensure that, in passing from $f_{n-1}(x)$ to $f_n(x)$ (and, consequently, in passing from $f_{n-1}(x)$ to $f_m(x)$, $m > n$), the lines of contact of the spherical pieces of the surface $f_{n-1}(\tau_3)$ remain fixed.

By virtue of the compactness of the class of K -quasiconformal mappings, from the sequence $f_n(x)$ one can extract a subsequence converging uniformly to a K -quasiconformal mapping $f(x)$ of the ball $|x| < 1$ onto itself, which maps τ_3 to a surface not quadrable in the sense of Lebesgue in any of its parts.

From Theorem 5, Definition 1, and the classical theorems of C. B. Morrey on Lebesgue surface area, it follows that a spatial K -quasiconformal mapping (for any $K > 1$) need not be quasiconformal on some plane sections of the domain of definition.

Remark. Using the methods of the proof of Theorem 5, one can give a negative answer to the following hypothesis (5): under a spatial quasiconformal mapping of a domain D , the images of all closed balls from D have finite de Giorgi perimeters (6).

Indeed, analogously to what was done in the proof of Theorem 5, for any number $K > 1$ one can construct a sequence $\{f_n(x)\}$ of K -quasiconformal mappings of the ball $|x| < 1$ onto itself, posse-

with the following properties: f_n , the image of the sphere $|x| = 1/2$, is a piecewise-spherical surface with area $\sigma_n > n$; in passing from $f_n(x)$ to $f_{n+1}(x)$, a portion of the surface $f_n(\{|x| = 1/2\})$, having area not exceeding 1, is sub-

jected to deformation; in passing from $f_n(x)$ to $f_m(x)$, $m > n + 1$, this same portion of the surface $f_n(\{|x| = 1/2\})$ is deformed. The limiting mapping is K -quasiconformal and maps the ball $|x| \leq 1/2$ onto a domain $f(\{|x| \leq 1/2\})$ with infinite perimeter in the sense of De Giorgi.

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