

THE INTEGRAL FUNCTIONAL AND ITS NORM

V. A. Gusev

1966

SovietRxiv

View the original and related papers at <https://soviextrxiv.org/items/ru-196601.00743>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.512.6

MATHEMATICS

V. A. GUSEV

THE INTEGRAL FUNCTIONAL AND ITS NORM

(Presented by Academician S. N. Bernstein on 25 VIII 1965)

In the monograph by E. V. Voronovskaya ⁽¹⁾, on p. 177, a number of still unsolved problems were formulated. Among them is the question of investigating the integral functional

$$1, \quad \frac{\xi}{2}, \quad \frac{\xi^2}{3}, \dots, \frac{\xi^n}{n+1} \quad \text{for } -\infty < \xi < +\infty. \quad (1)$$

In the present paper a functional of a more general form is investigated:

$$a_0, a_1\xi, a_2\xi^2, \dots, a_n\xi^n \quad \text{for } -\infty < \xi < +\infty, \quad (2)$$

where $(a_i)_0^n$ is an absolutely monotone segment ⁽¹⁾, p. 25). The functional (2), which we shall denote by F_ξ , may also be called an integral functional, since the numbers $(a_i)_0^n$ are the moments of some nondecreasing function $g(t)$ defined on the segment $[0, 1]$. Indeed,

$$F_\xi(x^i) = a_i\xi^i = \xi^i \int_0^1 t^i dg(t) = \int_0^\xi x^i dg\left(\frac{x}{\xi}\right) \quad (i = 0, 1, 2, \dots, n),$$

where $\xi t = x$ has been put, and if

$$P_n(x) = \sum_{i=1}^n p_i x^i$$

is an algebraic polynomial of degree not exceeding n , then

$$F_\xi[P_n(x)] = \int_0^\xi P_n(x) dg\left(\frac{x}{\xi}\right). \quad (3)$$

It is obvious that the functional (1) is a particular case of the functional (2), when $g(t) \equiv t$.

Theorem 1. *If the segment $(a_i)_0^n$ is amorphous ((¹), p. 27), then the integral functional (2) is an absolutely monotone segment for $0 \leq \xi \leq \xi_0$, where*

$$\xi_0 = \begin{cases} 1/x_m, & \text{if } n = 2m - 1, \\ 1/y_m, & \text{if } n = 2m. \end{cases}$$

Here x_m is the largest root of the orthogonal polynomial $X_m(x)$ corresponding to the integral weight $g(t)$, and y_m is the largest root of the orthogonal polynomial $Y_m(x)$, corresponding to the integral weight

$$\int_0^t \tau dg(\tau)$$

((²), p. 39).

Proof. 1) Let $n = 2m - 1$. We compute determinants of the form

$$\Delta(\alpha_i \xi, \alpha_{2k+1} \xi^{2k+1}) = |\alpha_{i+j+1} \xi^{i+j+1}|_{i,j=0}^k \quad (k = 0, 1, \dots, m-1),$$

$$\Delta(\alpha_0 - \alpha_1 \xi, \alpha_{2k} \xi^{2k} - \alpha_{2k+1} \xi^{2k+1}) = |\alpha_{i+j} \xi^{i+j} - \alpha_{i+j+1} \xi^{i+j+1}|_{i,j=0}^k$$

$$(k = 0, 1, \dots, m-1).$$

Obviously,

$$\Delta(\alpha_1 \xi, \alpha_{2k+1} \xi^{2k+1}) = \Delta(\alpha_1, \alpha_{2k+1}) \xi^{(k+1)^2} \quad (k = 0, 1, \dots, m-1),$$

$$\Delta(\alpha_0 - \alpha_1 \xi, \alpha_{2k} \xi^{2k} - \alpha_{2k+1} \xi^{2k+1}) = \xi^{(k+1)^2} X_{k+1} \left(\frac{1}{\xi} \right) \quad (k = 0, 1, \dots, m-1),$$

where

$$X_{k+1}(x) = |\alpha_{i+j} x - \alpha_{i+j+1}|_{i,j=0}^k \quad (k = 0, 1, 2, \dots)$$

is a system of orthogonal polynomials on the segment $[0, 1]$, corresponding to the integral weight $g(t)$ ((²), p.40). Since $\Delta(\alpha_1, \Delta_{2k+1}) > 0$ ($k = 0, 1, \dots, m-1$)

((¹), p.31), and the largest root x_m of the polynomial $X_m(x)$ lies on the segment $[0, 1]$ to the right of all roots of the polynomials $X_{k+1}(x)$ ($k = 0, 1, \dots, m-2$) ((²), p.58), it follows that for $0 < \xi < 1/x_m$ all determinants $\Delta(\alpha_1\xi, \alpha_{2k+1}\xi^{2k+1}) > 0$ and $\Delta(\alpha_0 - \alpha_1\xi, \alpha_{2k}\xi^{2k} - \alpha_{2k+1}\xi^{2k+1}) > 0$ ($k = 0, 1, \dots, m-1$), and the segment (2) is an absolutely monotone amorphous segment ((¹), p.31). For $\xi = 1/x_m = \xi_0$ only one determinant

$$\Delta(\alpha_0 - \alpha_1\xi_0, \alpha_{2m-2}\xi_0^{2m-2} - \alpha_{2m-1}\xi_0^{2m-1}) = 0,$$

and, consequently, the segment (2) for $\xi = \xi_0$ is an absolutely monotone segment with a nodal structure ((¹), pp.22 and 31). For $\xi = 0$, the segment (2), obviously, is absolutely monotone with a nodal structure ((¹), p.28).

2) Let $n = 2m$. We compute determinants of the form

$$\Delta(\alpha_0, \alpha_{2k}\xi^{2k}) = |\alpha_{i+j}\xi^{i+j}|_{i,j=0}^k \quad (k = 0, 1, \dots, m),$$

$$\Delta(\alpha_1\xi - \alpha_2\xi^2, \alpha_{2k-1}\xi^{2k-1} - \alpha_{2k}\xi^{2k}) = |\alpha_{i+j-1}\xi^{i+j-1} - \alpha_{i+j}\xi^{i+j}|_{i,j=1}^k$$

$$(k = 1, 2, \dots, m).$$

Obviously,

$$\Delta(\alpha_0, \alpha_{2k}\xi^{2k}) = \Delta(\alpha_0, \alpha_{2k})\xi^{k(k+1)} \quad (k = 0, 1, \dots, m),$$

$$\Delta(\alpha_1\xi - \alpha_2\xi^2, \alpha_{2k-1}\xi^{2k-1} - \alpha_{2k}\xi^{2k}) = \xi^{k(k+1)}Y_k(1/\xi)$$

$$(k = 1, 2, \dots, m),$$

where

$$Y_k(x) = |\alpha_{i+j-1}x - \alpha_{i+j}|_{i,j=1}^k \quad (k = 1, 2, \dots)$$

is a system of orthogonal polynomials on the segment $[0, 1]$, corresponding to the integral weight

$$\int \tau dg(\tau)$$

((²), p.40). The rest of the proof is carried out analogously to case 1).

Corollary 1. The extremal polynomial $Q_n(x)$ ((¹), p.24) of the functional F_{ξ_0} has the form

$$Q_n(x) = \begin{cases} 1 + \frac{c\xi_0^{n+1}X_m^2(x/\xi_0)}{\Delta^2(\alpha_0, \alpha_{n-1})(x-1)}, & \text{if } n = 2m-1, \\ 1 + \frac{c\xi_0^n x Y_m^2(x/\xi_0)}{\Delta^2(\alpha_1, \alpha_{n-1})(x-1)}, & \text{if } n = 2m, \end{cases}$$

where the multiplier $c > 0$ is chosen so that

$$\max_{[0,1]} |Q_n(x)| = 1.$$

1) Let $n = 2m-1$. Since

$$\Delta(\alpha_0 - \alpha_1\xi_0, \alpha_{2m-2}\xi_0^{2m-2} - \alpha_{2m-1}\xi_0^{2m-1}) = 0,$$

then, according to the corollary on p. 31 ((¹), the functional F_{ξ_0} has an extremal polynomial $Q_n(x)$, whose resolvent

$$R_m(x) = \prod_{i=1}^m (x - \sigma_i),$$

where

$$0 < \sigma_1 < \sigma_2 < \dots < \sigma_m = 1,$$

satisfies the conditions

$$F_{\xi_0}[x^k R_m(x)] = \int_0^{\xi_0} [\text{unclear: integrand}](x) dg\left(\frac{x}{\xi_0}\right) = 0 \quad (k = 0, 1, \dots, m-1).$$

Consequently,

$$R_m(x) = \frac{\xi_0^m}{\Delta(\alpha_0, \alpha_{n-1})} X_m\left(\frac{x}{\xi_0}\right)$$

((²), p.41). Next we construct the extremal polynomial $Q_n(x)$ according to ((¹), p. 19, case III.

2) Let $n = 2m$. Since

$$\Delta(\alpha_1\xi_0 - \alpha_2\xi_0^2, \alpha_{2m-1}\xi_0^{2m-1} - \alpha_{2m}\xi_0^{2m}) = 0,$$

the functional F_{ξ_0} has an extremal polynomial $Q_n(x)$, whose resolvent

$$R_{m+1}(x) = \prod_{i=1}^{m+1} (x - \sigma_i),$$

where $0 = \sigma_1 < \sigma_2 < \dots < \sigma_{m+1} = 1$, satisfies the conditions

$$F_{\xi_0}[x^k R_{m+1}(x)] = \int_0^{\xi_0} x^k R_{m+1}(x) dg\left(\frac{x}{\xi_0}\right) = 0 \quad (k = 0, 1, \dots, m-1).$$

Consequently,

$$R_{m+1}(x) = \frac{\xi_0^m x}{\Delta(\alpha_1, \alpha_{n-1})} Y_m\left(\frac{x}{\xi_0}\right),$$

and the extremal polynomial $Q_n(x)$ is constructed according to ⁽¹⁾, p. 19, case IV.

Corollary 2. If $\max_{[0,1]} |P_n(x)| = 1$, then for $0 \leq \xi \leq \xi_0$ the inequality

$$\left| \int_0^\xi P_n(x) dg\left(\frac{x}{\xi}\right) \right| \leq \alpha_0 \quad (4)$$

holds.

Indeed, for $0 \leq \xi \leq \xi_0$ the norm $N(\xi)$ of segment (2) is α_0 (⁽¹⁾, p. 25), and therefore

$$|F_\xi(P_n(x))| \leq N(\xi) = \alpha_0,$$

which coincides with inequality (4) by virtue of (3).

Theorem 2. If the segment $(a_i)_0^n$ is amorphous, then the functional F_ξ for $\xi > \xi_0$ has the greatest possible number s of nodes with positive loads, namely: $s = m$, if $n = 2m - 1$, and $s = m + 1$, if $n = 2m$.

This theorem is a special case of Theorem 3 of E. V. Voronovskaya ⁽³⁾.

Corollary 1. The extremal polynomial $Q_n(x, \xi)$ of the functional F_ξ ($\xi > \xi_0$) has the form

$$Q_n(x, \xi) = \begin{cases} 1 - q_n(\xi)(1-x) \prod_{i=1}^{m-1} [x - \sigma_i(\xi)]^2, & \text{if } n = 2m - 1, \\ 1 - q_n(\xi)x(1-x) \prod_{i=2}^m [x - \sigma_i(\xi)]^2, & \text{if } n = 2m, \end{cases}$$

where the leading coefficient $q_n(\xi)$ is determined by the conditions

$$\max_{[0,1]} |Q_n(x, \xi)| = 1$$

and

$$Q_n(x_0, \xi) = -1$$

for some $x_0 \in [0, 1]$, while $\sigma_1(\xi), \sigma_2(\xi), \dots, \sigma_m(\xi)$ are the nodes with positive loads of the polynomial $Q_n(x, \xi)$.

Corollary 2. Segment (2), for $\xi > \xi_0$, is of class II ⁽¹⁾, p. 42, and, consequently, has a unique extremal polynomial. According to the theorem on continuous deformation ⁽¹⁾, p. 70, the coefficients and nodes of the extremal polynomial $Q_n(x, \xi)$ are continuous functions of the parameter ξ .

Corollary 3. All roots of the derivative $\frac{\partial}{\partial x} Q_n(x, \xi)$ are real, distinct, and lie inside the interval $(0, 1)$, i.e. the polynomial $Q_n(x, \xi)$ has an internal deformation ⁽¹⁾, p. 120).

Remark. The norm $N(\xi)$ of the integral functional F_ξ satisfies the inequality

$$N(\xi) \leq \sum_{k=0}^n \left| \frac{1}{R'_{n+1}(\tau_k)} \int_0^\xi \frac{R_{n+1}(x)}{x - \tau_k} dg\left(\frac{x}{\xi}\right) \right|,$$

where $R_{n+1}(x) = \prod_{i=0}^n (x - \tau_i)$ is the resolvent of the Chebyshev polynomial $T_n(x) = \cos n \arccos(2x - 1)$, and $(\tau_i)_0^n$ are its nodes. This remark is a special case of Theorem 13 ⁽¹⁾, p. 39), when $\mu_k = \alpha_k \xi^k$ ($k = 0, 1, \dots, n$) and $c_i = \tau_{i-1}$ ($i = 1, 2, \dots, n+1$). From this remark there follows the existence of a number $\xi_1 \geq \xi_0$ such that for all $\xi \geq \xi_1$, $Q_n(x, \xi) = T_n(x)$. The number ξ_1 may be defined as the least positive root of the equation

$$\int_0^\xi T_n(x) dg\left(\frac{x}{\xi}\right) = \sum_{k=0}^n \left| \frac{1}{R'_{n+1}(\tau_k)} \int_0^\xi \frac{R_{n+1}(x)}{x - \tau_k} dg\left(\frac{x}{\xi}\right) \right|.$$

We also note that $Q_n(x, \xi) = (-1)^n T_n(x)$ for $\xi < 0$, since the segment (2) in this case is alternating.

Theorem 3. The norm $N(\xi)$ of the segment

$$\xi, \frac{\xi^2}{2}, \dots, \frac{\xi^{n+1}}{n+1} \quad (5)$$

for $\xi_0 \leq \xi \leq \xi_1$ is a function convex downward.

Proof.

$$N(\xi + \Delta\xi) - 2N(\xi) + N(\xi - \Delta\xi) = \int_0^{\xi + \Delta\xi} Q_n(x, \xi + \Delta\xi) dx - 2 \int_0^\xi Q_n(x, \xi) dx + \int_0^{\xi - \Delta\xi} Q_n(x, \xi - \Delta\xi) dx >$$

$$> \int_0^{\xi+\Delta\xi} Q_n(x, \xi) dx - 2 \int_0^{\xi} Q_n(x, \xi) dx + \int_0^{\xi-\Delta\xi} Q_n(x, \xi) dx = \int_{\xi}^{\xi+\Delta\xi} Q_n(x, \xi) dx - \int_{\xi-\Delta\xi}^{\xi} Q_n(x, \xi) dx = [Q_n(x, \xi)]_{\xi-\Delta\xi}^{\xi+\Delta\xi}$$

$$(0 < \theta_1 < 1, \quad 0 < \theta_2 < 1),$$

since, according to Corollary 3 of Theorem 2, the polynomial $Q_n(x, \xi)$ for $x \geq 1$ is an increasing function of x . Thus, the inequality

$$N(\xi + \Delta\xi) - 2N(\xi) + N(\xi - \Delta\xi) > 0$$

has been proved, which is sufficient for downward convexity of the norm $N(\xi)$.

Corollary. The norm $N(\xi)$ of the segment (5), for $\xi_0 < \xi < \xi_1$, can be estimated by means of a linear function, namely

$$N(\xi) < \xi_0 + \frac{\xi - \xi_0}{\xi_1 - \xi_0} \left\{ \int_0^{\xi_1} T_n(x) dx - \xi_0 \right\}.$$

In conclusion, we give estimates of integrals of algebraic polynomials $P_n(x)$ satisfying the condition $\max_{[0,1]} |P_n(x)| = 1$:

$$\left| \int_0^{\xi} P_n(x) dx \right| \leq \begin{cases} \left| \int_0^{\xi} T_n(x) dx \right|, & \text{for } \xi < 0, \\ \xi, & \text{for } 0 \leq \xi \leq \xi_0, \\ \xi_0 + \frac{\xi - \xi_0}{\xi_1 - \xi_0} \left\{ \int_0^{\xi_1} T_n(x) dx - \xi_0 \right\}, & \text{for } \xi_0 < \xi < \xi_1, \\ \int_0^{\xi} T_n(x) dx, & \text{for } \xi \geq \xi_1. \end{cases}$$

Leningrad Institute
of Aviation Instrument Engineering

Received
25 VIII 1965

References

- ¹ E. V. Voronovskaya, *The Method of Functionals and Its Applications*, 1963.
- ² G. Szegő, *Orthogonal Polynomials*, 1962.
- ³ E. V. Voronovskaya, DAN, 166, No. 6 (1966).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.