

Boundary Values of Functions from Weighted Classes

Yu. S. Nikol'skii

1965

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.94347>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1965. Volume 164, No. 3

UDC 517.514

MATHEMATICS

Yu. S. NIKOLSKII

BOUNDARY VALUES OF FUNCTIONS FROM WEIGHTED CLASSES

(Presented by Academician I. N. Vekua on 27 V 1965)

In the present paper we consider weighted classes of functions defined in the half-space $x_n > 0$ of points $x = (x_1, \dots, x_n)$ of the n -dimensional space E_n . The properties of the traces of these functions on the $(n-1)$ -dimensional subspace E_{n-1} , defined by the equation $x_n = 0$, are studied.

Let r be a natural number; $1 < p < \infty$; E^+ the upper half-space of points with $x_n > 0$, and $\varphi = \varphi(\rho)$ ($\rho = \sqrt{x_1^2 + \dots + x_n^2}$) a positive function (weight) defined on E^+ ; let $f = f(x)$ be a function defined on E^+ together with its generalized derivatives up to order r inclusive. By definition, the function f belongs to the class $L_{p,\varphi}^{(r)}(E^+)$ or $W_{p,\varphi}^{(r)}(E^+)$, if for it the finite norm

$$\|f\|_{L_{p,\varphi}^{(r)}(E^+)} = \sum_{|k|=r} \left\| \frac{f^{(k)}(x)}{\varphi(\rho)} \right\|_{L_p(E^+)}$$

or

$$\|f\|_{W_{p,\varphi}^{(r)}(E^+)} = \|f\|_{L_{p,\varphi}^{(r)}(E^+)} + \|f\|_{L_p(\omega)}$$

has meaning, where $k = (k_1, \dots, k_n)$ is an integer vector ($k_i \geq 0$, $i = 1, \dots, n$), $|k| = k_1 + k_2 + \dots + k_n$, ω is the intersection of E^+ with the unit ball in E_n centered at the origin, and

$$\|f\|_{L_p(\omega)} = \left\{ \int_{\omega} |f|^p dx \right\}^{1/p}, \quad f^{(k)}(x) = \frac{\partial^{|k|} f(x)}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}.$$

For $\varphi = 1$, the classes $L_{p,\varphi}^{(r)}$ and $W_{p,\varphi}^{(r)}$ become the well-known classes $L_p^{(r)}$ and $W_p^{(r)}$ of S. L. Sobolev ⁽¹⁾.

Let l be a positive non-integer number and $l = \bar{l} + \alpha$, where $\bar{l}$ is an integer and $0 < \alpha < 1$. By definition, a function $\psi = \psi(x_1, \dots, x_{n-1})$ belongs to the class $L_{p,\varphi}^{(l)}(E_{n-1})$ or $W_{p,\varphi}^{(l)}(E_{n-1})$, if it is given on E_{n-1} together with its generalized derivatives up to order $\bar{l}$ inclusive and has the finite norm

$$\|\psi\|_{L_{p,\varphi}^{(l)}(E_{n-1})} = \sum_{|k|=\bar{l}} \sum_{i=1}^{n-1} \left\{ \int_0^\infty \frac{dh}{h^{1+\alpha p}} \left\| \frac{\Delta_i(\psi^{(k)}, h)}{\varphi(\bar{\rho} + h)} \right\|_{L_p(E_{n-1})}^p \right\}^{1/p}$$

or

$$\|\psi\|_{W_{p,\varphi}^{(l)}(E_{n-1})} = \|\psi\|_{L_p(\omega_*)} + \|\psi\|_{L_{p,\varphi}^{(l)}(E_{n-1})},$$

where $\Delta_i(\psi, h)$ ($i = 1, \dots, n-1$) denotes the first difference of the function $\psi = \psi(x_1, \dots, x_{n-1})$ with step h in the variable x_i ; $\bar{\rho} = |x_1| + \dots + |x_{n-1}|$, and $\omega_* = \omega E_{n-1}$. Let the weight function $\varphi = \varphi(t)$ on $[0, \infty)$ be continuous, positive, nonincreasing, and satisfy the inequality $\varphi(2t) \leq c\varphi(t)$

(for sufficiently large t), where c is a positive constant independent of t . Then the following theorems hold.

Theorem 1. If the function $f \in L_{p,\varphi}^{(r)}(E^+)$ or $W_{p,\varphi}^{(r)}(E^+)$, then for it the boundary function

$$\psi = \psi(x_1, \dots, x_{n-1}) = f|_{E_{n-1}}$$

has meaning, and respectively the embeddings* hold

$$L_{p,\varphi}^{(r)}(E^+) \rightarrow L_{p,\varphi}^{(r-1/p)}(E_{n-1}), \quad W_{p,\varphi}^{(r)}(E^+) \rightarrow W_{p,\varphi}^{(r-1/p)}(E_{n-1}). \quad (1)$$

Theorem 2. If the function $\psi \in L_{p,\varphi}^{(r-1/p)}(E_{n-1})$, $W_{p,\varphi}^{(r-1/p)}(E_{n-1})$, then there exists a function $f \in L_{p,\varphi}^{(r)}(E^+)$, $W_{p,\varphi}^{(r)}(E^+)$, defined on E^+ , such that

$$f|_{E_{n-1}} = \psi$$

and the embeddings hold

$$L_{p,\varphi}^{(r-1/p)}(E_{n-1}) \rightarrow L_{p,\varphi}^{(r)}(E^+), \quad W_{p,\varphi}^{(r-1/p)}(E_{n-1}) \rightarrow W_{p,\varphi}^{(r)}(E^+). \quad (2)$$

Theorem 3. Suppose that on E_{n-1} a system of functions is given

$$\psi_i \in L_{p,\varphi}^{(r-i-1/p)}(E_{n-1}), \quad W_{p,\varphi}^{(r-i-1/p)}(E_{n-1}) \quad (i = 0, 1, \dots, r-1).$$

Then there exists a function $f \in L_{p,\varphi}^{(r)}(E^+), W_{p,\varphi}^{(r)}(E^+)$, defined on E^+ , such that

$$\partial^i f / \partial x_n^i \big|_{E_{n-1}} = \psi_i \quad (i = 0, 1, \dots, r-1)$$

and the inequality is fulfilled

$$\|f\|_{L_{p,\varphi}^{(r)}(E^+)} \leq c \sum_{i=0}^{r-1} \|\psi_i\|_{L_{p,\varphi}^{(r-i-1/p)}(E_{n-1})}$$

and, respectively, the analogous inequality in which everywhere L is to be replaced by W .

In the present paper the investigations of L. D. Kudryavtsev ⁽²⁾ are developed. The classes considered here are generalizations of the weighted classes $L_{p,\alpha}^{(r)}, W_{p,\alpha}^{(r)}$ introduced by him ($\varphi = (1 + \rho)^\alpha$), for which he obtained direct and inverse embedding theorems. But our embedding theorems differ from the corresponding theorems of L. D. Kudryavtsev. Along with the class $L_{p,\alpha}^{(r)}(E^+)$ he also considers the class $L_{p,\alpha}^{(r)}(E_1^+)$ of functions defined on the strip $E_1^+ \{x : 0 < x_n \leq 1\}$. Moreover, he introduces on E_{n-1} fractional classes $\bar{L}_{p,\alpha}^{(r)}(E_{n-1})$ and proves that

$$L_{p,\alpha}^{(r)}(E^+) \rightarrow \bar{L}_{p,\alpha}^{(r-1/p)}(E_{n-1}) \rightarrow L_{p,\alpha}^{(r)}(E_1^+).$$

In our case, however, the inverse embedding theorem completely reverses the direct theorem (see (1) and (2)). We have also succeeded in showing that the fractional classes of L. D. Kudryavtsev $L_{p,\alpha}^{(r-1/p)}(E_{n-1})$ are not equivalent to $L_{p,\varphi}^{(r-1/p)}(E_{n-1})$ (for $\varphi = (1 + \rho)^\alpha, \alpha > 0$).

Embeddings for unweighted spaces W_p^r were studied in the works ^(1, 3, 4, 12-15), while embeddings of the spaces $L_p^{(r)}$, i.e. spaces in which the norm of the elements does not include the norm of the function itself in L_p , were studied in ^(2-4, 15).

* If a function f defined on E^+ belongs to the normed space Λ_1 , and its trace $\psi = f|_{E_{n-1}}$ belongs to the normed space Λ_2 and $\|\psi\|_{\Lambda_2} \leq c_1 \|f\|_{\Lambda_1}$, then, as usual, we write $\Lambda_1 \rightarrow \Lambda_2$. Conversely, $\Lambda_2 \rightarrow \Lambda_1$ means that there exists on E^+ a function $f \in \Lambda_1$ such that $f|_{E_{n-1}} = \psi$ and $\|f\|_{\Lambda_1} \leq c_2 \|\psi\|_{\Lambda_2}$, where c_1 and c_2 are constants independent of f, ψ .

Theorem 4. Let the weight function be $\varphi = \rho^{\alpha_0} \lambda(\rho)$ ($\alpha_0 = (n - p)/p$), $1 < p < \infty$, and suppose that $\lambda(\rho)$ satisfies the condition

$$\int_1^\infty \frac{dz}{z\chi(z)} < \infty, \quad \chi(z) = \left\{ \min_{1 \leq t < \infty} \left[\frac{\lambda(zt)}{\lambda(t)} \right]^p \right\}^{1/p}.$$

Then the embedding holds

$$W_{p,\varphi}^{(r)}(E^+) \rightarrow W_{p,(1+\rho)^k\varphi}^{(r-k)}(E^+), \quad k = 0, 1, \dots, r.$$

The last theorem is analogous to the first assertion of Theorem 1 of paper ⁽⁵⁾; the classes $W_{\rho,\varphi}^{(r)}$ considered there were defined somewhat differently than in the present paper.

Consider the differential equation

$$L(u) = \sum_{|k|, |l| \leq r} (-1)^{|l|} \frac{\partial^l}{\partial x^l} (a_{kl}(x) u^{(k)}(x)) = F \quad (3)$$

in the class $\mathfrak{M}$ of functions f belonging to $L_{2,\varphi}^{(r)}(E^+)$ and satisfying the boundary conditions

$$\partial^i f / \partial x_n^i \big|_{E_{n-1}} = \psi_i \quad (i = 0, 1, \dots, r-1),$$

where $\psi_i \in L_{2,\varphi}^{(r-i-1/2)}(E_{n-1})$ are prescribed functions. It is assumed that

$$a_{kl}(x) = a_{lk}(x), \quad |a_{kl}(x)| \leq M^2 / [(1 + \rho)^{r-\min(|k|, |l|)} \varphi(\rho)]^2,$$

$$\sum_{|k|, |l| \leq r} a_{kl} \xi_k \xi_l \geq \frac{\lambda}{[\varphi(\rho)]^2} \sum_{|k|=r} \xi_k^2,$$

where $\lambda > 0$ does not depend on x, ξ_k, ξ_l . The function F has the property that the norm

$$\sigma_F = \|F\|_{L_2^{(r)}(E^+)} = \sup_{\substack{\|v\|_{L_{2,\varphi}^{(r)}(E^+)} \leq 1 \\ v \in \mathfrak{M}_0}} |(F, v)|, \quad (4)$$

is finite, where

$$(F, v) = \int_{E^+} F \cdot v \, dx$$

and $\mathfrak{M}_0$ is the class of functions $f \in L_{2,\varphi}^{(r)}(E^+)$ having zero boundary functions $\psi_i = 0$ ($i = 0, 1, \dots, r-1$). In particular, if

$$\int_{E^+} (1 + \rho)^{2r} \varphi^2 F^2 \, dx < \infty,$$

then condition (4) is satisfied.

Relying essentially on Theorems 1, 3, and 4, we prove by the variational method the following theorems.

Theorem 5. In the class $\mathfrak{M}$ there exists, and moreover is unique, a generalized solution u of equation (3).

Theorem 6. For the generalized solution U of the boundary-value problem under consideration, the estimate

$$\|U\|_{W_{2,\varphi}^{(r)}(E^+)} \leq c \left\{ \sum_{i=0}^{r-1} \|\psi_i\|_{W_{2,\varphi}^{(r-i-1/2)}(E_{n-1})} + \sigma_F \right\},$$

holds, where c is a constant independent of the boundary functions ψ_i ($i = 0, 1, \dots, r-1$) and of the quantity σ_F .

If one requires the functions $a_{kl}(x)$ and $F(x)$ to be sufficiently smooth, then from known results (see, for example, (6–9)) it follows that the solution U has continuous partial derivatives on E^+ up to order $2r$ inclusive and becomes a classical solution of equation (3).

Theorem 7. *The classical solution u of equation (3) in the class $\mathfrak{M}$ is unique.*

The proof of the last theorem is based essentially on the fact that it is shown that the class $\mathfrak{M}_{00}$ is everywhere dense in the class $\mathfrak{M}_0$ in the sense of the metric

$$D(f) = \int_{E^+} \sum_{|k|=r} \left[\frac{f^{(k)}(x)}{\varphi(\rho)} \right]^2 dx,$$

where $\mathfrak{M}_{00}$ is the class of finite functions $f \in \mathfrak{M}_0$.

We note that these investigations develop the works of L. D. Kudryavtsev (10, 11), in which the variational method for solving the first boundary-value problem in the case of an unbounded domain was considered for self-adjoint elliptic equations of the second order.

In conclusion I express my deep gratitude to L. D. Kudryavtsev for posing the problem and for his constant attention.

Moscow
Institute of Physics and Technology

Received
19 V 1965

REFERENCES

- ¹ S. L. Sobolev, *Some Applications of Functional Analysis to Equations of Mathematical Physics*, L., 1950.
- ² L. D. Kudryavtsev, DAN, **153**, No. 3, 530 (1963).
- ³ V. P. Il' in, V. A. Solonnikov, DAN, **136**, No. 3, 538 (1961).
- ⁴ V. A. Solonnikov, DAN, **134**, No. 2, 282 (1960).
- ⁵ Yu. S. Nikol' skii, DAN, **162**, No. 3 (1965).
- ⁶ P. I. Lizorkin, DAN, **134**, No. 4, 761 (1960).
- ⁷ O. A. Ladyzhenskaya, DAN, **79**, No. 5, 723 (1951).
- ⁸ S. Agmon, A. Douglis, N. Nirenberg, *Estimates of solutions of elliptic equations near the boundary under general boundary conditions*, II, 1962.
- ⁹ O. A. Ladyzhenskaya, DAN, **120**, No. 5, 956 (1958).
- ¹⁰ L. D. Kudryavtsev, *On the first boundary-value problem for elliptic equations with coefficients decreasing at infinity*, Soviet-American Symposium on Partial Differential Equations, Novosibirsk, 1963.
- ¹¹ L. D. Kudryavtsev, DAN, **157**, No. 1, 45 (1964).
- ¹² V. I. Kondrashov, DAN, **48**, 563 (1945).
- ¹³ E. Gagliardo, Rend. Semin. matem. univ. Padova, **27**, 284 (1957).
- ¹⁴ L. I. Slobodetskii, DAN, **118**, No. 2, 243 (1958).
- ¹⁵ V. P. Il' in, Tr. Mat. inst. im. V. A. Steklova AN SSSR, **66**, 227 (1962).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.