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V. N. Zhigulev

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Abstract

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AERODYNAMICS

V. N. ZHIGULEV

ON THE EQUATIONS OF TURBULENT MOTIONS OF A GAS

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1. The modern theory of turbulence (¹⁻³) proceeds from the existence of a statistical connection between the macroscopic characteristics of a gas at different points of space; the quantities sought are equations of space-time evolution for averaged quantities

$$\bar{\Phi} = \int \Phi P_s(\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_s) du_1 du_2 \dots du_s \quad (1)$$

(formula (1.1) of the book (³)); it is assumed that the space-time evolution of the instantaneous value of the mass-averaged velocity at the point $\mathbf{q}_i - \mathbf{u}_i$ obeys the Navier–Stokes equations; P_s is the probability density of the instantaneous values $\mathbf{u}_i$ ($1 \leq i \leq s$) at the points $\mathbf{q}_i$. In this theory, substantial successes have been achieved, confirmed by experiment, especially in elucidating the laws of homogeneous turbulence. However, its further development is connected with the resolution of a number of fundamental difficulties, the chief of which are: a) there are no equations for the functions P_s , which must constitute the rigorous basis of the theory of turbulence; b) there are no rigorous grounds for using the Navier–Stokes equations to obtain equations for $\bar{\Phi}$; c) there are no rigorous indications that such a system of functions $\bar{\Phi}_i$ is complete for describing the turbulent field.

It seems to us most natural, at the present stage of development of the theory of turbulence, to turn to the statistical mechanics of a gas ensemble, based on the classical Liouville equation (for turbulent motions must be described by this equation), and to develop in the theory of turbulence a system of concepts replacing the definition of the mean (1) and relying on the concept of the phase space of a gas.

In what follows we assume that the system of quantities analogous to the system of quantities (1) and replacing (1) will be the following:

$$\bar{\Phi} = \int \Phi F^{(s)}(t, x_1, x_2, \dots, x_s) dc_1 dc_2 \dots dc_s; \quad (2)$$

$F^{(s)}$ are distribution functions of complexes of s particles, which are functions of x_i ($1 \leq i \leq s$)—the aggregate of $\mathbf{q}_i$ and $\mathbf{c}_i$ —the coordinates and velocities of the groups of particles under consideration; the averaged function depends on $t, x_1, \dots, x_s$; the quantity $\bar{\Phi}$ in expression (2), just as in expression (1), is a function of $t, \mathbf{q}_1, \dots, \mathbf{q}_s$.

The system of concepts (2) is more detailed than the system of concepts (1), for it operates with the statistical connection between groups of particles possessing, in the elements $d\mathbf{q}_i$ (near the points $\mathbf{q}_i$), velocities in the elements $d\mathbf{c}_i$ (near the velocities $\mathbf{c}_i$); the question of the choice of the scales $\mathbf{q}_i$ and $\mathbf{c}_i$ in specific cases must be settled.

As will be seen from what follows, the definition of the mean (2) has a macroscopic basis, although in comparison with definition (1) it leads to the necessity of studying, so to speak, “temperature” phenomena, which also leads to a complete system of determining functions. For the functions $F^{(s)}$ there are equations—these are the Bogolyubov equations ⁽⁴⁾.

2. The study of quantities of the type (2) at once brings us into a circle of new problems of the kinetic theory of gases. Since in the study of turbulent motions we proceed from the existence of a statistical connection between groups of particles, the usually studied solutions of the Liouville equation (we have in mind the fundamental works ^(4–8), which assume the presence of molecular chaos in macrospace ⁽⁹⁾), i.e.,

$$F^{(s)} = \prod_{1 \leq i \leq s} F^{(1)}(t, x_i), \quad (3)$$

will be insufficient for us. The question of constructing an apparatus that makes it possible to consider the statistical connection between groups of particles was the subject of our work ⁽⁹⁾.

The necessity of extending the class of solutions (3) in the consideration of turbulence is indicated, among other things, by the fact that the equations constructed under the assumption (3) do not have stable solutions when turbulence arises; and as a result of the instability there arises, as is supposed, a statistical connection between the macroscopic elements of the gas.

A natural extension of the class of solutions (3) is the assumption of the existence of a weak statistical connection between groups of gas particles, leading to the solution ⁽⁹⁾

$$\frac{m^{3s}}{v^s} F^{(s)} = \prod_{1 \leq i \leq s} f(t, x_i) + \sum_{1 \leq i < j \leq s} \sum g(t, x_i, x_j) \prod_{\substack{1 \leq k \leq s \\ (k \neq i \neq j)}} f(t, x_k). \quad (4)$$

The functions f and g satisfy the system of equations

$$\begin{aligned}
\frac{\partial f}{\partial t} + \mathbf{c}_1 \frac{\partial f}{\partial \mathbf{q}_1} &= \int \{f(x_1)f(x_2)\}_1 dP_{12} + \int \{g(x_1, x_2)\}_1 dP_{12}, \\
\frac{\partial g}{\partial t} + \mathbf{c}_1 \frac{\partial g}{\partial \mathbf{q}_1} + \mathbf{c}_2 \frac{\partial g}{\partial \mathbf{q}_2} &= \int \{f(x_1)g(x_3, x_2) + f(x_3)g(x_1, x_2)\}_1 dP_{13} + \\
&+ \int \{f(x_2)g(x_1, x_3) + f(x_3)g(x_1, x_2)\}_2 dP_{23}; \quad (5)
\end{aligned}$$

the operator

$$\{F(x_1, \dots, x_s)\}_i = F(x_1, \dots, x_{i-1}x'_i, x_{i+1}, \dots, x_{s-1}x'_s) - F(x_1, \dots, x_s);$$

x'_i, x'_s, x_i, x_s are sets of phase points characterized by the same coordinate $\mathbf{q}_i$; the velocity quantities $\mathbf{c}'_i, \mathbf{c}'_s, \mathbf{c}_i, \mathbf{c}_s$ are connected by the law of binary collision ⁽¹²⁾; $dP_{ij} = |\mathbf{c}_i - \mathbf{c}_j| b db d\varepsilon' d\mathbf{c}_j$; b is the impact parameter, ε' the meridional angle.

Equations (5) are the simplest model of turbulence formulated in terms of the aerodynamics of a rarefied gas.

In favor of this model is the circumstance that hydrodynamic instability leading to turbulence may lead, in the first instance, only to the occurrence of the function g .

3. Equations (5) possess the following transport equations:

$$\int \left(\frac{\partial f}{\partial t} + \mathbf{c}_1 \frac{\partial f}{\partial \mathbf{q}_1} \right) \psi_i d\mathbf{c}_1 = 0, \quad \iint \left(\frac{\partial g}{\partial t} + \mathbf{c}_1 \frac{\partial g}{\partial \mathbf{q}_1} + \mathbf{c}_2 \frac{\partial g}{\partial \mathbf{q}_2} \right) \chi_i d\mathbf{c}_1 d\mathbf{c}_2 = 0, \quad (6)$$

where the quantities ψ_i are the 5 invariants of a binary collision; the quantities χ_i are quadratic forms of the invariants ψ_i : $1, m\mathbf{c}_1, m\mathbf{c}_1^2/2, m\mathbf{c}_1 m\mathbf{c}_2, m\mathbf{c}_1(m\mathbf{c}_2^2/2), (m\mathbf{c}_1^2/2)(m\mathbf{c}_2^2/2)$ —in all, 15 scalar quantities. Thus, equations (5) possess 20 transport equations (6); these, ultimately, are the sought equations for quantities of the type (2).

The system of equations (6) can be continued as applied to the system of equations (4) of work ⁽⁹⁾; in this case the averaged quantity will be a cubic form of the invariants, a fourth-degree form, and so on, which corresponds to the generalization of the functional series of the type (4) of the present article.

4. Bearing in mind processes occurring on scales much greater than the mean free path, it is natural to seek an analogue of normal solutions

Hilbert' s method ⁽¹⁰⁾ for system (5). Namely, we seek solutions for f and g in the form

$$f = f^{(0)} + f^{(1)} + f^{(2)} + \dots, \quad g = g^{(0)} + g^{(1)} + g^{(2)} + \dots, \quad (7)$$

where each subsequent term is related to the preceding one in order as the Knudsen number; for the functions $f^{(i)}$ and $g^{(i)}$ we have the recurrent system of integral equations:

$$\begin{aligned} & \int \{f^{(0)}(x_1)f^{(0)}(x_2)\}_1 dP_{12} + \int \{g^{(0)}(x_1, x_2)\}_1 dP_{12} = 0, \\ & \int \{f^{(0)}(x_1)g^{(0)}(x_3, x_2) + f^{(0)}(x_3)g^{(0)}(x_1, x_2)\}_1 dP_{13} + \\ & + \int \{f^{(0)}(x_2)g^{(0)}(x_1, x_3) + f^{(0)}(x_3)g^{(0)}(x_1, x_0)\}_2 dP_{23} = 0; \end{aligned} \quad (8)$$

$$\begin{aligned} & \int \{f^{(0)}(x_1)f^{(1)}(x_2) + f^{(1)}(x_1)f^{(0)}(x_2)\}_1 dP_{12} + \int \{g^{(1)}(x_1, x_2)\}_1 dP_{12} = \frac{\partial f^{(0)}}{\partial t} + c_1 \frac{\partial f^{(0)}}{\partial q_1}, \\ & \int \{f^{(1)}(x_1)g^{(0)}(x_3, x_2) + f^{(0)}(x_1)g^{(1)}(x_3, x_2) + f^{(1)}(x_3)g^{(0)}(x_1, x_2) + \\ & + f^{(0)}(x_3)g^{(1)}(x_1, x_2)\}_1 dP_{13} + \int \{f^{(1)}(x_2)g^{(0)}(x_1, x_3) + f^{(0)}(x_2)g^{(1)}(x_1, x_3) + \\ & + f^{(1)}(x_3)g^{(0)}(x_1, x_2) + f^{(0)}(x_3)g^{(1)}(x_1, x_2)\}_2 dP_{23} = \\ & = \frac{\partial g^{(0)}}{\partial t} + c_1 \frac{\partial g^{(0)}}{\partial q_1} + c_2 \frac{\partial g^{(0)}}{\partial q_2}, \dots \end{aligned} \quad (9)$$

In solving equations (8), (9), etc., it is necessary to use the condition of smallness of the statistical correlation, which underlies model (5), and which states that the quantity $g(x_1, x_2)/f(x_1)f(x_2)$ is small.

5. A solution of the system of equations (8), symmetric in the variables x_1 and x_2 , is

$$\begin{aligned} g^{(0)} = & \frac{f_{01}^{(0)} f_{02}^{(0)}}{n_1 n_2} \left[W_{112} + W_{121} + W_{212} mC_1 + W_{221} mC_2 + \widetilde{W}_{312} : mC_1 mC_2 + \right. \\ & + \widetilde{W}_{321} : mC_2 mC_1 + W_{412} \frac{mC_1^2}{2} + W_{421} \frac{mC_2^2}{2} + W_{512} mC_1 \frac{mC_2^2}{2} + W_{521} mC_2 \frac{mC_1^2}{2} + \\ & \left. + (W_{612} + W_{621}) \frac{mC_1^2}{2} \frac{mC_2^2}{2} \right], \end{aligned} \quad (10)$$

$$f^{(0)} = f_{01}^{(0)} \left[1 + \widetilde{W}_{311} : (\Phi_{31} C_1 C_1 + \Phi_{32} U) + W_{511} \Phi_5 C_1 + W_{611} \Phi_6 \right], \quad (11)$$

where $f_0^{(0)}$ is the local Maxwellian distribution; $f_{01}^{(0)}, f_{02}^{(0)}$ are the function $f_0^{(0)}$ taken either at the point x_1 or at x_2 ; n is the particle density; W_1 is a scalar function of $t, \mathbf{q}_1, \mathbf{q}_2$, with $W_{112} = W_1(t, \mathbf{q}_1, \mathbf{q}_2)$, and $W_{121} = W_1(t, \mathbf{q}_2, \mathbf{q}_1)$, similarly for the second and third indices of all quantities W_i ; W_2 is a vector-function of $t, \mathbf{q}_1, \mathbf{q}_2$; $\widetilde{W}_3$ is a tensor of rank two, which is a function of $t, \mathbf{q}_1, \mathbf{q}_2$, with $W_{312\alpha\beta} = W_{321\beta\alpha}$; W_4 is a scalar function of $t, \mathbf{q}_1, \mathbf{q}_2$; W_5 is a vector-function of $t, \mathbf{q}_1, \mathbf{q}_2$; W_6 is a scalar function of $t, \mathbf{q}_1, \mathbf{q}_2$; $C_1 = c_1 - V_1, C_2 = c_2 - V_2^*$, where V is the macroscopic velocity of the gas; $\widetilde{W}_{311}$ denotes $\widetilde{W}_3(t, \mathbf{q}_1, \mathbf{q}_1)$; the quantities $\Phi_{31}, \Phi_{32}, \Phi_5, \Phi_6$ are functions of the temperature T at the point $\mathbf{q}_1$ and of the modulus of the velocity C_1 ; these quantities are solutions of an integral equation of the Fredholm type of the second kind, analogous to that encountered in Enskog's method ⁽¹¹⁾ in solving the Boltzmann equation.

The uniqueness of the solutions (10) and (11) of equations (8) has been proved.

* The quantities C_i should be understood by us as the velocities of pulsational and chaotic (thermal) motion; the function g is responsible for the relative share of both.

A necessary condition for the solvability of equations (9) is the satisfaction of the transport equations (6) after substituting into them $f = f^{(0)}$ and $g = g^{(0)}$; this assertion, the uniqueness of the solutions (10) and (11), and also the transport equations themselves were proved by applying integral relations of type (5.28) from Chapter III of the book ¹².

6. Thus, we obtain a closed system of 20 scalar equations (6) for 15 different scalar functions W_i (moments of the binary function g) and 5 ordinary functions—the density n , the mass-average velocity $\mathbf{V}$, and the temperature T (moments of the distribution function f).

The system of equations obtained in this way, in the first approximation, can be refined by solving the system of equations (9) for the functions $f^{(1)}$ and $g^{(1)}$; for large-scale turbulence $L_T \gg L\sqrt{\text{Kn}}$ (L_T is the scale of turbulence, L is the principal scale of the flow, Kn is the Knudsen number, of the same order of magnitude as the reciprocal Reynolds number), the equations of the first approximation will be the main term of these equations; however, for small-scale turbulence $L_T \lesssim L\sqrt{\text{Kn}}$, the contribution of the second approximation will be essential (there is a formal analogy with the boundary layer). The second approximation, i.e., the functions $f^{(1)}$ and $g^{(1)}$, describe special turbulent dissipation and, apparently, the processes of disappearance of correlations.

Central Aerohydrodynamic Institute
named after N. E. Zhukovsky

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Note: Figure translations are in progress. See original paper for figures.

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