

GENERALIZED EXTENSIONS OF UNBOUNDED OPERATORS

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Abstract

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MATHEMATICS

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GENERALIZED EXTENSIONS OF UNBOUNDED OPERATORS

(Presented by Academician A. N. Kolmogorov on 27 III 1965)

I. Let H_0 be a complete Hilbert space with scalar product $(f, g)_0$ and norm $\|f\|_0$. Suppose that in H_0 there is an everywhere dense linear set H_+ , which is a complete Hilbert space with respect to another scalar product $(f, g)_+$ and norm $\|f\|_+$. We shall assume that

$$\|f\|_0 \leq \|f\|_+ \quad (f \in H_+).$$

We shall call the space H_+ a space with positive* norm, and also a space of basic elements. We shall say that every antilinear functional $a(f)$ on H_+ is generated by a generalized element a , and we shall write the value of the functional on the element $f \in H_+$ as $a(f) = (a, f)_0$. In what follows we use the notation $\overline{(a, f)_0} = (f, a)_0$. Obviously, the totality of all generalized elements is a linear set. Since the space H_+ is Hilbert, every antilinear functional on H_+ can be represented in the form of a scalar product defined in H_+ , i.e.,

$$(a, f)_0 = (a^*, f)_+ = (y_a, f)_+ \quad (a^* = y_a, a^* \in H_+, f \in H_+). \quad (1)$$

Equality (1) generates a linear operator J , mapping the set of generalized elements into the space H_+ . The range of the operator is, obviously, all of H_+ .

We introduce in the set of generalized elements H_- a scalar product, putting

$$(a, \beta)_- = (y_a, y_\beta)_+ \quad (a, \beta \in H_-). \quad (2)$$

The set H_- with the scalar product (2) is a Hilbert space.

It follows from (2) that the operator J is isometric, mapping H_- into H_+ . Since

$$\|a\|_- = \sup_{f \in H_+} \frac{|(a, f)_0|}{\|f\|_+},$$

the inequality holds

$$\|f\|_- \leq \|f\|_0 \leq \|f\|_+.$$

Thus,

$$H_+ \subseteq H_0 \subseteq H_-.$$

We shall call the space H_- a space with negative norm, and also the space of generalized elements of the Hilbert space H_0 . It is not difficult to prove that H_+ is dense in H_- .

II. Let $\Omega(f, g)$ be a bilinear functional defined in H_+ . Then it is easy to show that

$$\Omega(f, g) = (Bf, g)_0 \quad (f, g \in H_+), \quad (3)$$

* In the exposition of Sec. I we mainly follow the paper of Yu. M. Berezanskii⁽²⁾.

where B is a bounded linear operator acting from H_+ into H_- , which is uniquely determined by the bilinear functional Ω .

Let B be an arbitrary bounded linear operator acting from H_+ into H_- . The expression $(f, Bg)_0$ ($f, g \in H_+$) is, obviously, a bilinear functional in H_+ . Then, according to (3), there is a uniquely determined bounded linear operator $B^\times$, mapping H_+ into H_- , for which

$$(f, Bg)_0 = (B^\times f, g)_0 \quad (f, g \in H_+).$$

We shall call the operator $B^\times$ the **generalized adjoint operator** with respect to B . If $B = B^\times$, then such an operator will be called a **generalized self-adjoint operator**.

III. Let T be a closed operator acting in the Hilbert space H_0 , for which

$$T_0 \subset T, \quad T^* \subset T_0^*,$$

where T_0 is a symmetric operator with dense domain of definition.* Consider the Hilbert space $H_+ = D_{T_0^*}$ with scalar product

$$(f, g)_+ = (T_0^* f, T_0^* g)_0 + (f, g)_0 \quad (f, g \in H_+).$$

Construct the triple of spaces $H_+ \subseteq H_0 \subseteq H_-$. We now pose the following problem: can the operators T and T^* be extended linearly to all of $H_+ = D_{T_0^*}$ so that the obtained extensions T_{H_+} and $T_{H_+}^\times$ are adjoint to one another in the generalized sense? It turns out that this can always be done; moreover, the obtained generalized extensions T_{H_+} and $T_{H_+}^\times$ will be linear bounded operators acting from H_+ into H_- , and, in contrast to T and T^* , will have the same domain of definition H_+ . In the paper a number of properties of generalized extensions are established, their resolvent is investigated, and an estimate for its norm in the space H_- is found.

IV. **Theorem 1.** *If the operator T belongs to the class Ω , then T and T^* can always be extended** to all of $H_+ = D_{T_0^*}$ so that the obtained extensions $T_{H_+}(H_+ \rightarrow H_-)$ and $T_{H_+}^\times(H_+ \rightarrow H_-)$ are adjoint to one another in the generalized sense.*

Definition. A linear extension T_{H_+} of the operator T to $H_+ = D_{T_0^*}$ will be called **generalized** if $T_{H_+}^\times$ is an extension of T^* to H_+ . It follows from Theorem 1 that if the operator T belongs to the class Ω , then it has a nontrivial generalized extension T_{H_+} , which, as examples show, is not determined uniquely by the operator. However, the following holds.

Theorem 2. Let T_{H_+} and T_{H_+}' be generalized extensions of the operator T to the space $H_+ = D_{T_0^*}$, and let

$$T_{H_+} = A + iB; \quad T_{H_+}' = A' + iB',$$

where A, B, A', B' are linear generalized self-adjoint operators acting from H_+ into H_- . Then, if $B = B'$, then $A = A'$.

We shall regard the generalized extension of the operator T as an operator acting in the Hilbert space H_- . This operator has in H_- the dense domain of definition H_+ .

Theorem 3. If λ is a regular point of an operator T belonging to the class Ω , then it is a point of regular type for any generalized extension $T_{H_+}(H_+ \rightarrow H_-)$ of the operator T .

* An operator T possessing the indicated properties will henceforth be assigned to the class Ω .

** In the theorem the existence is proved of a nontrivial generalized extension, different from that which is obtained if T and T^* are extended by zero.

Corollary 1. If $\tilde{T}_{H_+}$ is an arbitrary generalized extension of the operator T , then at every point λ regular for T there exists the resolvent $(T_{H_+} - \lambda I)^{-1}$, whose closure is a bounded operator defined everywhere in H , and for which

$$\|(\overline{T_{H_+} - \lambda I})^{-1}\| \leq (1 + |\lambda|) (1 + \|(T - \lambda I)^{-1}\|_0). \quad (4)$$

Corollary 2. If the number λ is not a point of regular type for some generalized extension T_{H_+} , then it belongs to the spectrum of the operator T .

Corollary 3. If the operator T has no spectrum in the finite part of the plane, then every number λ is a point of regular type for every generalized extension T_{H_+} , and, consequently, in this case there exists the resolvent $(T_{H_+} - \lambda I)^{-1}$, whose closure is a bounded operator in H , and for which (4) holds.

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Note: Figure translations are in progress. See original paper for figures.

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