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Abstract

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MATHEMATICS

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ON THE METHOD OF AVERAGING IN SYSTEMS OF INTEGRO-DIFFERENTIAL EQUATIONS

(Presented by Academician N. N. Bogolyubov on 13 IV 1965)

The method of averaging for systems of differential equations was developed in the works of N. N. Bogolyubov ^(1,2), Yu. A. Mitropolsky ^(2,3), V. M. Volosov ⁽⁴⁾, and other authors.

The present paper is devoted to extending the method of averaging to systems of integro-differential equations of the form

$$\frac{dx}{dt} = \varepsilon X(t, x) + \varepsilon \int_0^t Z(t, x(s), s) ds, \quad (1)$$

where $\varepsilon > 0$ is a small parameter; $x = \{x_1, \dots, x_n\}$ is an n -dimensional vector; $X(t, x)$, $Z(t, x, s)$ are real vector-functions, defined and continuous for all t and s from $[0, +\infty)$ and for all $x \in E_n$ (E_n is n -dimensional Euclidean space).

1. Suppose that the limits exist

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t, x) dt = X_0(x), \quad \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s Z(t, x, s) ds = Z_0(t, x).$$

Along with equation (1), consider the averaged equation

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) + \varepsilon \int_0^t Z_0(t, \xi(s)) ds. \quad (2)$$

Concerning the closeness of the solutions of equations (1) and (2), the following can be proved.

Theorem. *Suppose that the functions $X(t, x)$ and $Z(t, x, s)$ satisfy the following conditions:*

1) For some domain $D \subset E_n$ one can indicate constants M , N , λ , and μ such that, for all $t \geq 0$, $s \geq 0$, and for any points x, x', x'' from D , the inequalities

$$|X(t, x)| \leq M, \quad |X(t, x') - X(t, x'')| \leq \lambda|x' - x''|,$$

$$|Z(t, x, s)| \leq N, \quad |Z(t, x', s) - Z(t, x'', s)| \leq \mu|x' - x''|$$

hold.

2) Uniformly with respect to $x \in D$ and t , the limits exist

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T X(t, x) dt = X_0(x), \quad \lim_{s \rightarrow \infty} \frac{1}{s} \int_0^s Z(t, x, s) ds = Z_0(t, x).$$

Then, for arbitrarily small positive η , ρ , and for arbitrarily large $L > 0$, one can assign such a positive ε_0 that, if $\xi = \xi(t)$ is a solution of equation (2), defined on the interval $0 \leq t < \infty$ and lying

lying in the domain D together with its ρ -neighborhood, then for $0 \leq \varepsilon < \varepsilon_0$ on the interval $0 < t < Le^{-k}$, $0 < k < 1/2$, the inequality

$$|x(t) - \xi(t)| < \eta, \quad (3)$$

holds, where $x(t)$ is the solution of equation (1) coinciding with $\xi(t)$ at $t = 0$.

Proof. Following the method of N. N. Bogolyubov ⁽¹⁾, introduce the function

$$u(t, x) = \int_D \Delta_a(x - x') \left\{ \int_0^t [X(\tau, x') - X_0(x')] d\tau + \right. \\ \left. + \int_0^t \left[\int_0^\tau (Z(\tau, x', s) - Z_0(\tau, x')) ds \right] d\tau \right\} dx',$$

where

$$\Delta_a(x) = \begin{cases} A_a \left(1 - \frac{|x|^2}{a^2} \right)^2, & |x| \leq a, \\ 0, & |x| > a, \end{cases} \quad \int_{E_n} \Delta_a(x) = 1.$$

Using the conditions of the theorem, it is not difficult to establish the following inequalities:

$$|u(t, x)| \leq tf(t) + t^2\psi(t), \quad \left| \frac{\partial u(t, x)}{\partial x} \right| \leq I_a[tf(t) + t^2\psi(t)],$$

$$\left| \frac{\partial u(t, x)}{\partial t} - X(t, x) + X_0(x) - \int_0^t [Z(t, x, s) - Z_0(t, x)] ds \right| \leq 2a(\lambda + \mu t),$$

$$I_a = \int_{E_n} \left| \frac{\partial \Delta_a}{\partial x} \right| dx,$$

where $f(t)$ and $\psi(t)$ are monotonically decreasing functions tending to zero as $t \rightarrow \infty$.

Let us now form the expression

$$R = \frac{d\bar{x}}{dt} - \varepsilon X(t, \bar{x}) - \varepsilon \int_0^t Z(t, \bar{x}, s) ds$$

and compute it for $\bar{x} = \xi(t) + \varepsilon u(t, \xi(t))$, where $\xi(t)$ is the solution of equation (2), defined on the interval $0 \leq t < +\infty$ and lying in D with its ρ -neighborhood. Carrying out the corresponding calculations, we find

$$\begin{aligned} |R(t)| \leq & 2a\varepsilon(\lambda + \mu t) + 4N\varepsilon t + I_a \varepsilon^2 [tf(t) + t^2\psi(t)](M + Nt) + \\ & + \lambda \varepsilon^2 [tf(t) + t^2\psi(t)] + \mu \varepsilon^2 [t^2\alpha(t) + t^3\beta(t)]. \end{aligned} \quad (4)$$

Here $\alpha(t) \rightarrow 0$, $\beta(t) \rightarrow 0$ as $t \rightarrow \infty$.

Let $x(t)$ be the solution of equation (1) coinciding with $\xi(t)$ at $t = 0$. Then on the interval $0 < t < t^*$, $t^* < Le^{-k}$, on which $x(t) \in D$, we have

$$\left| \frac{d(\bar{x} - x)}{dt} \right| \leq \lambda \varepsilon |\bar{x} - x| + \mu \varepsilon \int_0^t |\bar{x}(s) - x(s)| ds + |R(t)|.$$

Consequently,

$$|\bar{x} - x| \leq \left| \frac{1}{k_2 - k_1} \int_0^t |R(\tau)| [k_2 e^{k_2(t-\tau)} - k_1 e^{k_1(t-\tau)}] d\tau \right|, \quad (5)$$

$$k_2 = \frac{\lambda \varepsilon}{2} + \sqrt{\mu \varepsilon + \left(\frac{\lambda \varepsilon}{2} \right)^2}, \quad k_1 = \frac{\lambda \varepsilon}{2} - \sqrt{\mu \varepsilon + \left(\frac{\lambda \varepsilon}{2} \right)^2}.$$

Now, using the properties of the functions α, β, f, ψ and disposing of the parameters a and ε_0 , on the basis of inequalities (4) and (5), we find that on the interval $0 < t < t^*$, $0 \leq \varepsilon < \varepsilon_0$, inequality (3) will hold. It is easy to show that $t^* = Le^{-k}$. The theorem is proved.

2. If the function $Z(t, x, s)$ is averaged not with respect to s , but with respect to t , i.e., if one assumes the existence of the limit

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T Z(t, x, s) dt = Z_{01}(x, s),$$

then we obtain the averaged integro-differential equation of the form

$$\frac{d\xi}{dt} = \varepsilon X_0(\xi) + \varepsilon \int_0^t Z_{01}(\xi(s), s) ds,$$

which reduces to the system of differential equations

$$\frac{d\xi}{dt} = \eta, \quad \frac{d\eta}{dt} = \varepsilon \frac{\partial X_0(\xi)}{\partial \xi} \eta + \varepsilon Z_{01}(\xi, t).$$

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CITED LITERATURE

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- ³ Yu. A. Mitropolsky, *Problems of the Asymptotic Theory of Nonstationary Oscillations*, 1964.
- ⁴ V. M. Volosov, UMN, **17**, 6, 3 (1962).

Note: Figure translations are in progress. See original paper for figures.

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