

ON ISOINVOLUTIVE DECOMPOSITIONS OF LIE ALGEBRAS

L. V. Sabinin

1965

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196501.84819>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 519.4

MATHEMATICS

L. V. SABININ

ON ISOINVOLUTIVE DECOMPOSITIONS OF LIE ALGEBRAS

(Presented by Academician S. L. Sobolev on 13 IV 1965)

In the present paper, for compact real semisimple Lie algebras, the theory of isoinvolutive decompositions is set forth (without proofs) and its application to compact symmetric spaces of rank 1.

Let Γ_r be a compact semisimple Lie algebra over the field of real numbers; let $e_1, e_2, \dots, e_r$ be its basis. The structure is then determined by the commutators of the basis vectors

$$[e_I e_K] = c_{IK}^R e_R, \quad c_{IK}^R = -c_{KI}^R, \quad I, K, R = 1, \dots, r, \quad (1)$$

and by the Jacobi identities

$$[e_I [e_J e_K]] + [e_K [e_I e_J]] + [e_J [e_K e_I]] = 0. \quad (2)$$

The transformations of the adjoint group Γ_r^* have the form

$$\bar{\eta} \rightarrow e^{\bar{c}(u)} \bar{\eta}, \quad \bar{c}(u) = \|c_{JK}^I u^K\|, \quad (3)$$

and the tensor c_{IK}^R is invariant under such transformations.

In a compact semisimple Lie algebra one can introduce a positive definite metric tensor

$$a_{IJ} = c_{IK}^P c_{PJ}^K, \quad a_{IJ} = a_{JI} \quad (4)$$

and consider orthonormal bases, in which

$$a_{IJ} = \delta_{IJ}. \quad (5)$$

Then, as is known,

$$c_{JK}^I = -c_{IK}^J. \quad (6)$$

An automorphism A of the algebra Γ is called an **invomorphism** if

$$A^2 = I, \quad A \neq I. \quad (7)$$

In some orthobasis any invomorphism S has a diagonal form with ± 1 on the main diagonal.

All $\eta \in \Gamma_r$ such that $S\eta = \eta$ form a subalgebra (invoalgebra) $L \subset \Gamma_r$; the pair Γ_r/L will be called an **invopair**. Obviously,

$$\Gamma_r = E + L, \quad E \perp L; \quad S\bar{\eta} = \begin{cases} \bar{\eta}, & \bar{\eta} \in L, \\ -\bar{\eta}, & \bar{\eta} \in E. \end{cases} \quad (8)$$

The invopair Γ/L generates the symmetric space Γ^*/L^* , where L^* has on E an irreducible linear representation, if Γ is simple.

Definition 1. We shall say that the algebra Γ is an **involutive sum of its subalgebras** $\Gamma = L_1 + L_2 + L_3$, if L_1, L_2, L_3 are the invoalgebras of commuting involomorphisms $S_1, S_2, S_3 = S_1 S_2$, respectively.

Obviously, $L_1 \cap L_2 = L_1 \cap L_3 = L_2 \cap L_3 = L_0$, and L_i/L_0 ($i = 1, 2, 3$) are invopairs.

Definition 2. An involutive sum $\Gamma = L_1 + L_2 + L_3$ will be called **isoinvolutive** if L_1 and L_2 are conjugate in Γ^* , with the conjugating automorphism of the form $e^{\bar{c}(\bar{\xi}t)}$, where $\bar{\xi} \in E_3 = L_3 - L_0$.

Lemma 1. If Γ/L is a compact semisimple involpara, then there exists $\bar{\xi} \in E = \Gamma - L$ such that the subgroup $e^{c(\bar{\xi}t)}$ in Γ^* is compact.

Corollary 1. If Γ is a semisimple compact Lie algebra and S is its involomorphism, then there exist inner automorphisms Q_n in Γ^* (n a positive integer) such that $(Q_n)^n = 1$, $SQ_n S = Q_n^{-1}$.

On the basis of Lemma 1 and its corollary the following theorem is proved:

Theorem 1. Let Γ be a semisimple compact Lie algebra, and let L_1 be any of its invoalgebras; then there exists an isoinvolutive decomposition

$$\Gamma = L_1 + L_2 + L_3, \quad L_1 = \varphi L_2, \quad \varphi \in \Gamma^*, \quad \varphi^4 = 1, \quad S_3 = \varphi^2.$$

Corollary 1. The restriction of φ to L_3 acts as an involmorphism or as the identity automorphism.

Corollary 2. The restriction of φ to L_0 acts as an involmorphism or the identity automorphism.

Corollary 3. *If the restriction of φ to L_3 acts as an involmorphism, then L_3 is an isoinvolutive sum $L_3 = L_0 + L'_0 + L''_0$, where L_0, L'_0, L''_0 are the involalgebras of the involmorphisms $S_1, (\varphi^{1/2})^{-1}S_1(\varphi^{1/2}), \varphi$, respectively.*

Definition 3. An isoinvolutive sum will be called **of type I** if the restriction of the conjugating automorphism φ to L_3 is the identity automorphism.

We shall consider only simple compact algebras Γ .

Let $\Pi = L_0 \cap L'_0$; then

$$\begin{aligned} L_0 &= \Pi + E_0, & L'_0 &= \Pi + E'_0, & L''_0 &= \Pi + E''_0, \\ L_1 &= L_0 + E_1, & L_2 &= L_0 + E_2, \end{aligned}$$

where E, E', E'' are orthogonal to Π , and E_1, E_2 are orthogonal to L_0 . From Theorem 1 and its corollaries we have

$$E_2 = \varphi E_1, \quad E'_0 = (\varphi^{1/2})E_0, \quad (\varphi^{1/2})\Pi = \Pi, \quad (\varphi^{1/2})E''_0 = E''_0. \quad (9)$$

Choosing orthonormal bases in Π, E_1, E_0, E''_0 , with the help of (9) we construct bases in E_2 and E'_0 . The union of all the indicated bases generates an orthobasis in Γ , which we shall call **isoinvolutive**. Let, in this notation,

$$\left. \begin{aligned} X_{i_1} \quad (i = 1, \dots, n) & \text{ be a basis in } E_1, \\ X_{i_2} \quad (i = 1, \dots, n) & \text{ be a basis in } E_2; \\ Y_{\alpha_2} \quad (\alpha = 1, \dots, \rho) & \text{ be a basis in } E_0, \\ Y_{\alpha_1} \quad (\alpha = 1, \dots, r) & \text{ be a basis in } \Pi; \end{aligned} \right\} Y_\alpha \quad (10)$$

$$\left. \begin{aligned} V_{\alpha_2} \quad (\alpha = 1, \dots, \rho) & \text{ be a basis in } E'_0, \\ V_{\alpha_1} \quad (a = 1, \dots, m) & \text{ be a basis in } E''_0. \end{aligned} \right\} V_\alpha$$

By construction

$$\begin{aligned} \varphi X_{i_1} &= X_{i_2}, & \varphi X_{i_2} &= -X_{i_1}, & \varphi Y_\alpha &= \varphi^\beta_\alpha Y_\beta \\ & & (\varphi Y_{\alpha_2} &= -Y_{\alpha_2}, & \varphi Y_{\alpha_1} &= Y_{\alpha_1}), \\ \varphi V_\alpha &= \varphi^b_\alpha V_b & (\varphi V_{\alpha_2} &= -V_{\alpha_2}, & \varphi V_{\alpha_1} &= V_{\alpha_1}). \end{aligned} \quad (11)$$

The automorphism $\theta = \varphi^{1/2}$ acts, as is easily verified, as follows:

$$\begin{aligned} \theta X_{i_1} &= \frac{1}{\sqrt{2}} \theta^j_i (X_{j_1} + X_{j_2}), & \theta X_{i_2} &= \frac{1}{\sqrt{2}} \theta^j_i (-X_{j_1} + X_{j_2}), \\ \theta^p_k \theta^k_s &= \delta^p_s, & \theta^i_j &= \theta^j_i, \end{aligned}$$

$$\theta Y_{a_2} = V_{a_2}, \quad \theta V_{a_2} = -Y_{a_2}, \quad \theta Y_{a_1} = \theta_{a_1}^{\beta_1} Y_{\beta_1}, \quad \theta V_{a_1} = \theta_{a_1}^{c_1} V_{c_1},$$

$$\theta_{a_1}^{\lambda_1} \theta_{\mu_1}^{a_1} = \delta_{\mu_1}^{\lambda_1}, \quad \theta_{a_1}^{c_1} \theta_{b_1}^{a_1} = \delta_{b_1}^{c_1}, \quad \theta_{\alpha_1}^{\beta_1} = \theta_{\beta_1}^{\alpha_1}, \quad \theta_{a_1}^{c_1} = \theta_{c_1}^{a_1}. \quad (12)$$

Taking into account the action of the automorphisms S_1, S_2, S_3 , we write the structure of the algebra Γ in the form

$$[X_{i_1} X_{j_1}] = -b_{i_1 j_1}^a Y_a, \quad [X_{i_2} X_{j_2}] = -b_{i_2 j_2}^a Y_a, \quad [X_{i_1} X_{j_2}] = q_{i_1 j_2}^a V_a,$$

$$[X_{i_1} V_a] = t_{i_1 a}^{j_2} X_{j_2}, \quad [X_{p_2} V_a] = t_{p_2 a}^{j_1} X_{j_1}, \quad [V_a V_c] = -b_{ac}^\alpha Y_\alpha, \quad [X_{i_1} Y_\alpha] = a_{i_1 \alpha}^{p_1} X_{p_1}, \quad (13)$$

$$[X_{i_2} Y_\alpha] = a_{i_2 \alpha}^{p_2} X_{p_2}, \quad [V_c Y_\alpha] = a_{c \alpha}^b V_b, \quad [Y_\alpha Y_\beta] = c_{\alpha \beta}^\gamma Y_\gamma.$$

Introduce the notation

$$a_{k_1 \alpha}^{p_1} = a_{k \alpha}^p, \quad b_{k_1 j_1}^a = b_{k j}^a, \quad q_{k_2 j_1}^a = q_{k j}^a, \quad t_{i_1 a}^{p_2} = t_{i a}^p. \quad (14)$$

Then the automorphism φ from (11), for the structure (13), gives

$$\begin{aligned} b_{i_2 j_2}^\alpha &= \varphi_\beta^\alpha b_{ij}^\beta, \quad q_{ij}^b = \varphi_a^b q_{ji}^a, \quad t_{i_2 b}^{j_1} = -\varphi_b^a t_{ia}^j, \quad a_{i_2 \alpha}^{p_2} = \varphi_\alpha^\beta a_{i \beta}^p, \\ b_{ac}^\alpha &= \varphi_a^f \varphi_c^d \varphi_\beta^\alpha b_{fd}^\beta, \quad a_{c \alpha}^b = \varphi_c^d \varphi_f^b \varphi_\alpha^\beta a_{d \beta}^f, \quad c_{\alpha \beta}^\gamma = \varphi_\alpha^\lambda \varphi_\beta^\mu \varphi_\nu^\gamma c_{\lambda \mu}^\nu. \end{aligned} \quad (15)$$

From the orthonormality of the isoinvobasis it also follows that

$$a_{j \alpha}^i = b_{ji}^\alpha, \quad q_{kj}^a = t_{ja}^k, \quad t_{i_2 b}^{j_1} = -t_{jb}^i, \quad a_{c \alpha}^f = b_{cf}^\alpha, \quad c_{\beta \gamma}^\alpha = -c_{\beta \gamma}^\alpha. \quad (16)$$

The automorphism θ from (12) for the structure (13) leads to the relations

$$\begin{aligned} t_{i \alpha_2}^p &= \theta_i^p \theta_i^k a_{k \alpha_2}^s, \quad \theta_i^k \theta_s^b t_{ia}^p = \theta_a^b t_{kb}^s, \quad \theta_i^k \theta_s^p a_{k \beta_1}^s = \theta_{\beta_1}^{\alpha_1} a_{i \alpha_1}^p, \\ c_{\alpha_2 \beta_2}^{\lambda_1} &= -b_{\alpha_2 \beta_2}^{\sigma_2} \theta_{\sigma_1}^{\lambda_1}, \quad b_{\alpha_1 \beta_2}^{\sigma_2} = \theta_{\alpha_1}^{c_1} a_{c_1 \beta_2}^{\sigma_2}, \quad a_{\alpha_2 c_2}^{e_1} = \theta_{b_1}^{e_1} a_{c_2 \alpha_2}^{b_1}, \quad a_{e_2 \alpha_1}^{b_2} = \theta_{\alpha_1}^{\beta_1} c_{e_2 \beta_1}^{b_2}, \end{aligned} \quad (17)$$

$$a_{c_1 \alpha_1}^{b_1} \theta_{b_1}^{e_1} = \theta_{c_1}^{d_1} \theta_{\alpha_1}^{\beta_1} a_{d_1 \beta_1}^{e_1}, \quad c_{\alpha_1 \beta_1}^{\lambda_1} \theta_{\lambda_1}^{\mu_1} = \theta_{\alpha_1}^{\sigma_1} \theta_{\beta_1}^{\nu_1} c_{\sigma_1 \nu_1}^{\mu_1}.$$

Finally, consideration of all Jacobi identities leads to the relations

$$\begin{aligned}
a_{k\alpha}^p b_{ij}^\alpha &= 2a_{k\alpha_2}^p b_{ij}^{\alpha_2} + t_{j\alpha_2}^p q_{ki}^{\alpha_2} - t_{i\alpha_2}^p q_{kj}^{\alpha_2} + t_{j\alpha_1}^p q_{ki}^{\alpha_1} - t_{i\alpha_1}^p q_{kj}^{\alpha_1}, \\
-b_{\alpha_1 c_1}^{b_1} a_{i\alpha_1}^p &= t_{ja}^p t_{ic_1}^j - t_{jc_1}^p t_{ia_1}^j, \\
-a_{c_2\alpha_2}^{b_1} t_{ib_1}^k &= t_{pc_2}^k a_{i\alpha_2}^p + a_{p\alpha_2}^k t_{ic_2}^p, \\
a_{c_1\alpha_2}^{b_2} t_{ib_2}^k &= t_{pc_1}^k a_{i\alpha_2}^p + a_{p\alpha_2}^k t_{ic_1}^p, \\
a_{c_1\alpha_1}^{b_1} t_{ib_1}^k &= a_{p\alpha_1}^k t_{ic_1}^p - t_{pc_1}^k a_{i\alpha_1}^p, \\
c_{\alpha_2\beta_2}^{\sigma_1} a_{j\sigma_1}^l &= a_{i\beta_2}^l a_{j\alpha_2}^i - a_{i\alpha_2}^l a_{j\beta_2}^i, \\
c_{\beta_2\alpha_1}^{\sigma_2} a_{j\sigma_2}^l &= a_{i\beta_2}^l a_{j\alpha_1}^i - a_{i\alpha_1}^l a_{j\beta_2}^i, \\
c_{\alpha_1\beta_1}^{\sigma_1} a_{j\sigma_1}^l &= a_{i\beta_1}^l a_{j\alpha_1}^i - a_{i\alpha_1}^l a_{j\beta_1}^i.
\end{aligned} \tag{18}$$

Definition 4. An isoinvolutive sum, if it is not of type I, shall be called **type II** if $\theta_s^p = +\delta_s^p$, and **type III** otherwise.

The use of the relations obtained above leads to the theorem:

Theorem 2. If a compact simple Lie algebra Γ admits an isoinvolutive decomposition

$$\Gamma = L_1 + L_2 + L_3$$

of type I or II, then the maximal subalgebra of elements fixed under the action of the conjugating auto-

morphism φ , has center $\xi \in L_3 - L_0$. Hence it follows that, if the involara Γ/L_1 has rank 1, then $L_3 - L_0$ is one-dimensional for type I, and $L_0'' - \Pi$ is one-dimensional for type II.

This permits one to obtain the following theorems:

Theorem 3. If the isoinvolutive sum of type I

$$\Gamma = L_1 + L_2 + L_3$$

is simple and compact, and Γ/L_1 is an involara I, then Γ/L_1 is of the form

$$SO(n+1)/SO(n),$$

where L_1/L_0 is of the form

$$SO(n)/SO(n-1)$$

(with the natural embedding).

Theorem 4. If Γ/L_1 is a simple compact involara of rank 1, and

$$\Gamma = L_1 + L_2 + L_3$$

is an isoinvolutive decomposition of type II and E_0 is one-dimensional, then Γ/L_1 is of the form

$$SU(n+1)/S(U(n) \times \tilde{U}(1)),$$

and $L_1/\tilde{L}_0$ is of the form

$$SU(n)/S(U(n-1) \times U(1))$$

(with the natural embeddings)

$$(L_0 - \tilde{L}_0 = Z$$

is a one-dimensional center in L_1).

Theorem 5. If Γ/L_1 is a simple compact involara of rank 1,

$$\Gamma = L_1 + L_2 + L_3$$

is an isoinvolutive decomposition of type II and E_0 is three-dimensional, then Γ/L_1 is of the form

$$Sp(n+1)/Sp(n) \times Sp(1),$$

and $L_1/\tilde{L}_0$ is of the form

$$Sp(n)/Sp(n-1) \times Sp(1)$$

(with the natural embeddings)

$$(L_0 - \tilde{L}_0 = Z$$

is a three-dimensional ideal in L_1).

Institute of Mathematics

Siberian Branch of the Academy of Sciences of the USSR

Received

13 IV 1965

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.