

ON ESTIMATING THE NUMBER OF SOLUTIONS OF CERTAIN EQUATIONS

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Abstract

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MATHEMATICS

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ON ESTIMATING THE NUMBER OF SOLUTIONS OF CERTAIN EQUATIONS

(Presented by Academician Yu. V. Linnik on March 23, 1965)

Consider the equation

$$x_1^n + \dots + x_k^n = y_1^n + \dots + y_k^n, \quad (1)$$

where $P > \exp n^6$, $1 \leq x_i, y_i \leq P$, $i = 1, 2, \dots, k$. Let $I_{k,n}(P)$ be the number of solutions of this equation. An upper estimate for $I_{k,n}(P)$ is closely connected with the asymptotic formula in Waring's problem (see, for example, ⁽⁴⁾, p. 111). By the method he created, I. M. Vinogradov obtained, for $I_{k,n}(P)$ with $k \geq 4n^2 \ln n$, the estimate

$$I_{k,n}(P) \leq C(k, n)P^{2k-n}, \quad (2)$$

where $C(k, n)$ is a constant depending only on n and k .

Let us note that

$$I_{k,n}(P) \geq (2k)^{-1}P^{2k-n}. \quad (3)$$

Trivially we have

$$I_{k,n}(P) \leq P^{2k-1}.$$

It follows from the Hardy-Littlewood theorem that for $k \geq 2$ one has

$$I_{k,n}(P) \leq P^{2k-2+\varepsilon}.$$

The known problem of an asymptotic formula in Waring's problem reduces to obtaining the estimate (2) for $k \geq 4n + 1$ or $k \geq n^{1+\varepsilon}$ (see ^(2, 4)). In this article the question of estimates of type (2) is studied for possibly smaller k .

In what follows we shall use the following notation: $k, m, n, r, s, t, l, x, y, \lambda, P$ are integers; $n \geq 2$; $2 \leq r \leq n$; $I_{k,n}(P)$ is the number of solutions of equation (1); $c, c_1, c_2, \dots$ are constants that may depend on n and k ; $A, A_1, A_2, \dots$ are absolute constants.

Theorem 1. For $k \geq 6rn \ln n$, the inequality

$$I_{k,n}(P) \leq c_1 P^{2k-r} \quad (4)$$

holds.

Theorem 2. Let $1 \leq m \leq n$. Then, for $k \geq 6mn \ln n$, we have

$$I_{k,n}(P) \leq c_2 I_{k,m}(P). \quad (5)$$

Theorems 1 and 2 are easily proved from the following main theorem.

Main theorem. Let $p > \exp n^6$ be a prime number and let p_i be prime numbers satisfying the inequalities:

$$\frac{1}{4} p_i^{1-1/r} < p_{i+1} < \frac{1}{2} p_i^{1-1/r}, \quad i = 1, 2, \dots, \tau = [5r \ln n]; \quad p_1 = p.$$

Let, further, $q_1 = p_1^r \dots p_\tau^r$ and $c_3 q_1^{1/r} \leq P \leq c_4 q_1^{1/r}$. Consider the system of congruences

$$\begin{aligned} x_1 + \dots - y_k &\equiv \lambda_1 \\ &\vdots \\ x_1^n + \dots - y_k^n &\equiv \lambda_n \end{aligned} \pmod{q_1},$$

$$1 \leq x_i, y_i \leq P, \quad i = 1, 2, \dots, k.$$

If $N_k^{(q_1)}(\lambda_1, \dots, \lambda_n)$ is the number of solutions of this system of congruences, then for $k \geq 6rn \ln n$ the estimate holds

$$N_k^{(q_1)}(\lambda_1, \dots, \lambda_n) \leq N_k^{(q_1)}(0, \dots, 0) \leq c_5 P^{2k-rn+r(r-1)/2}.$$

Proof of Theorem 1. Consider the congruence

$$x_1^n + \dots + y_k^n \equiv 0 \pmod{q}, \quad 1 \leq x_i, y_i \leq P, \quad i = 1, 2, \dots, k,$$

and let $N_{kn}^{(q)}(P)$ be the number of solutions of this congruence. Obviously,

$$I_{k,n}(P) \leq N_{kn}^{(q)}(P).$$

Further we have

$$N_{k,n}^{(q)}(P) = \sum_{\lambda_1, \dots, \lambda_{n-1}} N_{k,n}^{(q)}(\lambda_1, \dots, \lambda_{n-1}, 0) \leq N_{k,n}^{(q)}(0, \dots, 0) \sum_{\lambda_1, \dots, \lambda_{n-1}} 1.$$

Putting $q = q_1$ and applying the main theorem, we obtain the required result.

Proof of Theorem 2. Using inequality (4), for $k \geq 6mn \ln n$ we have

$$I_{k,n}(P) \leq c_6 P^{2k-m}.$$

For $I_{k,m}(P)$ it is easy to obtain a lower estimate, for any k , of the form

$$I_{k,m}(P) \geq c_7 P^{2k-m}.$$

The assertion of the theorem follows from these two inequalities.

Inequalities of the type (4) and (5) also hold for systems of equations.

Theorem 3. Consider the system of equations

$$\begin{aligned} x_1^{m_1} + \dots - y_k^{m_1} &= 0, \\ &\dots\dots\dots \\ x_1^{m_s} + \dots - y_k^{m_s} &= 0, \\ x_1^n + \dots - y_k^n &= 0, \\ 1 \leq x_i, y_i \leq P, \quad i &= 1, 2, \dots, k, \\ 1 \leq m_1 < m_2 < \dots < m_s < m_{s+1} = n, \quad 1 \leq s \leq n. \end{aligned} \tag{6}$$

Let r be an arbitrary integer, $1 \leq r \leq n$, and let the integer t be determined by the inequalities

$$m_t \leq r < m_{t+1}.$$

Then, for $k \geq 6rn \ln n$, for the number of solutions I'_k of this system the estimate

$$I'_k \leq c P^{2k-\delta}, \quad \text{where } \delta = m_1 + \dots + m_t + (s-t+1)r,$$

holds.

Theorem 4. Along with the system (6), consider the system of equations:

$$\begin{aligned} x_1^{n_1} + \dots - y_k^{n_1} &= 0, \\ &\dots\dots\dots \\ x_1^{n_l} + \dots - y_k^{n_l} &= 0, \\ 1 \leq x_i, y_i \leq P, \quad i &= 1, 2, \dots, k, \\ 1 \leq n_1 < n_2 < \dots < n_l \leq n, \end{aligned}$$

and let I''_k be the number of solutions of this system. Define the smallest r by the inequalities

$$m_t \leq r < m_{t+1}, \quad n_1 + \dots + n_l \leq m_1 + \dots + m_t + r(s-t+1).$$

Then for $k \geq 6rn \ln n$ the relation

$$I'_k \leq c_8 I''_k$$

holds.

Theorems 3 and 4 are proved analogously to Theorems 1 and 2.

By complicating the proof, one can sharpen the estimates obtained above.

We note that a somewhat weaker assertion than Theorem 1 follows from my paper (5).

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Note: Figure translations are in progress. See original paper for figures.

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