

Construction of the Complete Set of Solutions to the Homogeneous Dirichlet Problem for the Vibrating String Equation

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Abstract

Full Text

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MATHEMATICS

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CONSTRUCTION OF THE COMPLETE SET OF SOLUTIONS OF THE HOMOGENEOUS DIRICHLET PROBLEM FOR THE EQUATION OF VIBRATIONS OF A STRING

(Presented by Academician S. L. Sobolev on 19 I 1965)

1. It is known ⁽¹⁾ that the operator Q , acting in the Hilbert space of functions $u(x, y) \in \dot{W}_2^{(1)}$ by the formula

$$Qu = \Delta^{-1}(\partial^2/\partial y^2 - \partial^2/\partial x^2)u,$$

where Δ^{-1} is the operator inverse to the Laplace operator under zero boundary conditions, is a bounded self-adjoint operator whose eigenfunctions are solutions of the homogeneous boundary-value problem

$$L_\lambda(u) = (1 + \lambda)\partial^2 u/\partial x^2 - (1 - \lambda)\partial^2 u/\partial y^2 = 0, \quad |\lambda| < 1; \quad (1)$$

$$u|_\Gamma = 0, \quad (2)$$

considered in a domain D with piecewise smooth boundary Γ . It is also known ⁽²⁾ that the spectrum of the operator Q depends qualitatively on the form of the boundary and, being, as a rule, of countable multiplicity, may be purely point spectrum or may contain intervals of continuity.

From what has been said above it follows ⁽³⁾ that, in the study of spectral expansions generated by the operator Q for somewhat arbitrary domains D , one cannot restrict oneself to classical solutions of problem (1), (2), or even to generalized solutions belonging to the space $\dot{W}_2^{(1)}(D)$.

2. In our papers (see, for example, ⁽³⁾) a certain class of piecewise constant functions was constructed, representing generalized solutions of the boundary-value problem (1), (2), or eigenfunctionals of the operator Q . In the present note it is established that every generalized solution (eigenfunctional) isomorphic to a piecewise continuous function is the limit of a uniformly convergent sequence of linear combinations of the above-mentioned piecewise constant eigenfunctionals.

Below we shall use the concepts and notation introduced in ⁽⁴⁾.

Lemma 1. Let T_λ be an eigenfunctional of the boundary-value problem (1), (2), isomorphic to a summable function $u_\lambda(x, y)$ in D . Then

$$u_\lambda(x, y) = f(y - \mu x) + g(y + \mu x), \quad \mu = \sqrt{(1 - \lambda)/(1 + \lambda)}, \quad (3)$$

where f and g are summable on the corresponding intervals and $u_\lambda(x, y)$ vanishes almost everywhere on Γ .

Conversely, if a function $u_\lambda(x, y)$, summable in D , is representable in the form (3) and vanishes almost everywhere on Γ , then

$$\iint_D u_\lambda(x, y) [(1 + \lambda)\varphi_{xx} - (1 - \lambda)\varphi_{yy}] dx dy = 0 \quad \varphi \in \Phi_0, \quad (4)$$

i.e.

$$T_\lambda(\varphi) = \iint_D u_\lambda(x, y) \varphi(x, y) dx dy$$

is an eigenfunctional of the boundary-value problem (1), (2).

We shall not prove this lemma here, but merely note that part of our reasoning is based on the integral identity

$$\iint_D u(x, y) \frac{\partial^2 \varphi}{\partial x \partial y} dx dy = \int_\Gamma u \frac{\partial \varphi}{\partial x} \cos ny ds + \int_\Gamma g \frac{\partial \varphi}{\partial s} ds.$$

II. 3. If an “admissible” domain D is fixed, then for each prescribed $\lambda = \lambda_0$ only two mutually exclusive cases are possible:

A. The set $A(\lambda_0, \Gamma)$ is empty. Such a value of the parameter λ_0 , as well as the automorphism S_{λ_0} itself, will be called **ergodic**. In this case it can be proved that if the set of points on the boundary Γ is invariant with respect to the automorphism S_{λ_0} , then it has either measure zero or full measure.

B. The set $A(\lambda_0, \Gamma)$ is a nonempty closed set on Γ , and consequently its complement consists of no more than a countable number of open arcs:

$$CA(\lambda_0, \Gamma) = \bigcup_{(k)} d_k(\theta'_k, \theta''_k).$$

Such a value λ_0 and the automorphism S_{λ_0} will be called **nonergodic**. Obviously, $A(\lambda_0, \Gamma)$ is invariant with respect to the automorphism S_{λ_0} , and together with each point θ it contains the whole set $\mathfrak{M}(\lambda_0, \theta, \Gamma)$.

It is easy to see that $S_{\lambda_0}^{\tau_0} d_k = d_k$ ($k = 1, 2, \dots$), i.e. each of these arcs is invariant with respect to one and the same iteration of the automorphism S_{λ_0} ; therefore these arcs will be called **adjacent periodic arcs**.

Let us agree, furthermore, to call two points (respectively, two arcs) λ -distinct if they do not belong to one and the same λ -cycle $\mathfrak{M}(\lambda, \theta, \Gamma)$ (respectively,

$\mathfrak{M}(\lambda, d, \Gamma)$). Let $\hat{A}(\lambda_0, \Gamma)$ be such a collection of λ_0 -distinct points that from $\theta \in A(\lambda_0, \Gamma)$ it follows that one of the points

$$S_{\lambda_0}^k \theta, \quad S_{\lambda_0}^k S_{\lambda_0} \theta \quad (k = 0, 1, 2, \dots, r_0 - 1)$$

belongs to $\hat{A}(\lambda_0, \Gamma)$. Let U_λ be the collection of discontinuous solutions $u(x, y, \lambda, \theta)$, constructed earlier (see (3)), of the homogeneous boundary-value problem (1), (2) for fixed λ and all $\theta \in \hat{A}(\lambda, \Gamma)$. Obviously, for ergodic values of the parameter λ the set U_λ is empty.

Lemma 2. Let θ_k ($k = 1, 2, \dots, N$) be a finite set of distinct points belonging to $\hat{A}(\lambda, \Gamma)$. Then the functions $u(x, y, \lambda, \theta_k) \in U_\lambda$ corresponding to these points are linearly independent.

Lemma 3. Let T_λ be an eigenfunctional of the boundary-value problem (1), (2), isomorphic to some piecewise-constant function. Then this functional is isomorphic to a finite linear combination of functions from U_{λ_0} .

Relying on the Poincaré-Denjoy theory of orientation-preserving topological mappings of closed curves onto themselves, one can prove the following auxiliary proposition, essential for what follows.

Lemma 4. Let T_λ be an eigenfunctional corresponding to a nonergodic value of the parameter λ , and let it be isomorphic to a piecewise-continuous function $v_\lambda(x, y)$. Then in representation (3) of this function each of the components f and g is constant on each of the adjacent periodic arcs, and these constants are identical for λ -indistinguishable arcs.

We shall now consider the collection of centers of adjacent periodic arcs and choose from them only one representative from each set of λ_0 -indistinguishable points; then we adjoin to this set the previously constructed set ...

the set $A(\lambda_0, \Gamma)$. The set $M(\lambda_0, \Gamma)$ obtained in this way, which, as is shown below, plays a central role in the study of the homogeneous Dirichlet problem for the string equation, will be called the **generating set of boundary points**.

In the case when the value λ_0 is ergodic, $A(\lambda_0, \Gamma)$ is empty, the adjacent arc is unique and coincides with the entire boundary; therefore a single point (no matter which) serves as the generating set. We also note that in the nonergodic case any two generating sets can be made to coincide if in them certain points are replaced by other λ_0 -indistinguishable points.

Let us consider one simplest example. Let D be the disk $x^2 + y^2 < 1$; then the automorphism S_λ turns out to be ergodic if the ratio α/π is irrational, and nonergodic otherwise (here $\operatorname{tg} \alpha = \sqrt{(1-\lambda)/(1+\lambda)}$ is the slope of the characteristics of equation (1)). In the nonergodic case there are no adjacent arcs, since $A(\lambda, \Gamma) = \Gamma$; therefore $M(\lambda, \Gamma) = \hat{A}(\lambda, \Gamma)$, and if, for example, $\alpha = \pi/2n$, then the arc $|\pi/2 - \theta| < \pi/2n$ may serve as a generating set.

A function f , defined on Γ , is called **invariant with respect to the automorphism** S_λ^+ if $f(S_\lambda^+\theta) = f(\theta)$ for all $\theta \in \Gamma$.

Lemma 5. Let the piecewise-continuous function f given on Γ be invariant with respect to the automorphisms $S_{\lambda_0}^+, S_{\lambda_0}^-$, and vanish on some generating set $M(\lambda_0, \Gamma)$. Then it is equal to zero on the entire boundary Γ .

Lemma 6. Let $F(\theta)$ be an arbitrary piecewise-continuous function given on Γ , and let $M(\lambda_0, \Gamma)$ be some generating set. Then there exists a piecewise-continuous function $f(\theta)$, invariant with respect to both automorphisms $S_{\lambda_0}^+, S_{\lambda_0}^-$, and coinciding with $F(\theta)$ on all of $M(\lambda_0, \Gamma)$.

Thus, any function invariant with respect to the automorphisms $S_{\lambda_0}^+, S_{\lambda_0}^-$ is completely determined by its values only on the generating set (see Lemma 5), and, on the other hand, its values on this set may be arbitrary (see Lemma 6).

Lemma 7. If f is an arbitrary measurable function on Γ , invariant with respect to the automorphisms $S_{\lambda_0}^+, S_{\lambda_0}^-$ for some ergodic value of the parameter λ_0 , then f is almost everywhere on Γ equal to some constant.

We note further that the structure of the generating set $M(\lambda_0, \Gamma)$ depends not only on the form of the boundary of the domain and on the value of the parameter λ , but also on the class of generalized solutions of problem (1), (2) under consideration. The generating set $M(\lambda_0, \Gamma)$ constructed above corresponds to the class of piecewise-continuous functions. If, for example, the class of solutions considered is narrowed to the class of continuous functions, then it can be proved that the generating set will be $\hat{A}(\lambda_0, \Gamma)$ (nothing need be prescribed on adjacent periodic arcs, since everything is determined by continuity).

If, conversely, the class of solutions considered is enlarged so that their components in representation (3) have, at each point on Γ , only one-sided limiting values, then it can be proved* that the generating set will again be $M(\lambda_0, \Gamma)$.

Lemma 8. Let the piecewise-continuous function f given on Γ be invariant with respect to the automorphisms $S_{\lambda_0}^+, S_{\lambda_0}^-$, and let its modulus on the generating set $M(\lambda_0, \Gamma)$ not exceed ε . Then $|f| \leq \varepsilon$ on the entire boundary Γ .

* We do not dwell on the details here.

Let $v(x, y, \lambda)$ be piecewise continuous in the domain D , vanish on Γ , and be representable in the form (3). Then each of its components f and g , considered as a function of the point θ on Γ , will be invariant with respect to both automorphisms S_λ^+, S_λ^- ; therefore the lemmas formulated above make it possible to establish the validity of the following main theorem.

Theorem. Let the value of the parameter λ_0 be nonergodic, and let T_{λ_0} be an arbitrary eigenfunctional isomorphic to the piecewise continuous function $v(x, y, \lambda_0)$. Then, for any $\varepsilon > 0$,

$$\max_{(x,y) \in \bar{D}} \left| v(x, y, \lambda_0) - \sum_{k=1}^N a_k u(x, y, \lambda_0, \theta_k) \right| < \varepsilon,$$

where a_k are certain constants, and $u(x, y, \lambda_0, \theta_k) \in U_{\lambda_0}$.

Thus, it turns out that the entire manifold of piecewise continuous generalized eigenfunctions is exhausted by linear combinations (or by their limits in the sense of uniform convergence in the closed domain) of the piecewise constant eigenfunctions constructed earlier.

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