

Some Questions in Approximation Theory

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Abstract

Full Text

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SOME QUESTIONS IN APPROXIMATION THEORY

(Presented by Academician A. N. Kolmogorov on 20 VII 1964)

MATHEMATICS

I. In the present note we shall deal mainly with four problems of approximation theory. Let X be a real Banach space with unit ball U , and let L_n be its finite-dimensional subspace.

Problem 1. Approximation of individual functions by a fixed finite-dimensional subspace. The problem consists in finding the quantity

$$E_{L_n}(f_0) = \inf_{\varphi \in L_n} \|f_0 - \varphi\|$$

of the least deviation of an element $f_0 \in X$ from L_n , and a polynomial $\varphi_0 \in L_n$ of best approximation, for which

$$E_{L_n}(f_0) = \|f_0 - \varphi_0\|.$$

Problem 2. Approximation of convex sets by finite-dimensional subspaces. Let F be a convex set lying in X . The quantity

$$\delta(F, L_n) = \sup_{f \in F} E_{L_n}(f)$$

will be called the deviation of the set F from L_n . An element $f_0 \in F$ will be called extremal if

$$\delta(F, L_n) = E_{L_n}(f_0).$$

The problem consists in determining the deviation $\delta(F, L_n)$ and the extremal elements.

Problem 3. Widths of sets. It is required to find the quantity

$$d_n(F, X) = \inf_{L_n} \delta(F, L_n) = \inf_{L_n} \inf_{\varepsilon} (F \subset L_n + \varepsilon U),$$

called the n -width of F .

Problem 4. Find

$$d^n(F) = \inf_{L^n} \inf_{\varepsilon} \{F \cap L^n \subset \varepsilon U \cap L^n\},$$

where L^n is a subspace of codimension n , i.e. a subspace of X consisting of elements f for which

$$f_1^*(f) = \dots = f_n^*(f) = 0,$$

and f_i^* are linearly independent elements of X^* . The quantity $d_n(F, X)$ was introduced by A. N. Kolmogorov. The usefulness of considering the quantity $d^n(F)$ was indicated by I. M. Gelfand.

The principal concrete results of the note concern the space C_Q of continuous real functions on a topological bicomact space Q , with metric

$$\|f\| = \max_{x \in Q} |f(x)|.$$

Sometimes the space C_Q^B on Q , with values in a Banach space B , and with norm

$$\|f\| = \max_Q \|f(x)\|_B$$

will be considered.

Problem 1 has been investigated in considerable detail from the general point of view in the works of Singer (see, for example, ⁽¹⁾), so that the following theorem is probably only of methodological interest, and we give it for the sake of unity of exposition.

Theorem 1. In order that an element $\varphi_0 \in L_n$ be a polynomial of best approximation to a function $f(x) \in C_Q^B$, it is necessary and sufficient that

there are $n+1$ points $x_1, \dots, x_{n+1}$ and $n+1$ functionals $f_1^*, \dots, f_{n+1}^* \in B^*$ such that

$$\begin{aligned} \sum_{k=1}^{n+1} \|f_k^*\| &= 1, & \sum_{k=1}^{n+1} f_k^*(f_k(x)) &= E_{L_n}(f_0), \\ \sum_{k=1}^{n+1} f_k^*(\varphi(x_k)) &= 0 & \text{for all } \varphi(x) \in L_n. \end{aligned} \quad (1)$$

From Theorem 1 there automatically follow all the criteria known to the author for polynomials of best approximation: those of Chebyshev ⁽²⁾, Kolmogorov ⁽³⁾, Remez ⁽⁴⁾, Zukhovitskii and Krein ⁽⁵⁾, Zukhovitskii and Stechkin ⁽⁶⁾.

The proof of Theorem 1 is extremely simple—it is a combination of the finite-dimensional Hahn–Banach theorem and the following generalization of Shnirelman's lemma ⁽⁷⁾.

In Q there exists a system of $n+1$ points $x_1, \dots, x_{n+1}$ at which the least deviation of the function $f_0(x)$ from L_n coincides with the least deviation of $f_0(x)$ from L_n on all of Q .

Theorem 1, in a somewhat different form, is contained in ⁽¹⁾.

We pass to problem 2. It belongs to the class of minimax problems for which, apparently, there is not yet a sufficiently developed analytic apparatus. Some initial approaches to problem 2 are contained in a joint work of A. A. Milyutin and the author. Here we shall give one inconclusive necessary condition for an extremal element in the space C_Q^B .

Theorem 2. For the element $f_0(x)$ to be extremal in the set F , it is necessary that there exist $n+1$ points $x_1, \dots, x_{n+1} \in Q$ and functionals f_k^* , satisfying the conditions (1), for which, in addition, the inequality

$$\sum_{k=1}^{n+1} f_k^*(f_0(x_k) - f(x)) \geq 0 \quad \text{for all } f \in F \quad (2)$$

holds.

In the case C_Q , the computation of $\delta(F, L_n)$ reduces to the finite-dimensional problem of finding

$$\max_{f \in F} \max \sum_{k=1}^{n+1} \mu_k f(x_k), \quad (2')$$

where the first max is taken over all points $x_1, \dots, x_{n+1}$ and all real collections $\mu_1, \dots, \mu_{n+1}$ for which

$$\sum_{k=1}^{n+1} |\mu_k| = 1, \quad \sum_{k=1}^{n+1} \beta_k \varphi(x_k) = 0 \quad \text{for all } \varphi \in L_n.$$

It seems to us interesting to make a more detailed investigation of problem (2'), in particular, to determine what the standard methods of the calculus of variations can give here.

We pass to the problem on the widths d_n . Let us also introduce the value of the *linear* width:

$$d'_n(F, X) = \inf_{L_n} \inf_{\Lambda: F \rightarrow L_n} \|f - \Lambda f\|,$$

where Λ is a linear operator mapping F into L_n .

Theorem 3. Among all Banach spaces, only Hilbert spaces possess the property that always

$$d_n(F; X) = d'_n(F, X) \geq d^n(F), \quad d_n(F, X) = d_n(F, X'), \quad F \in X \in X'.$$

The first examples of the noncoincidence of d_n and d'_n and of the decrease of d_n when F is immersed in an enveloping space were indicated in ⁽⁹⁾.

II. We now pass to some concrete results. Everywhere below the space X is $C_{[-1,1]}$.

Theorem 4. For any integers $n \geq 0$ and $r > 0$ there exists, and is uniquely determined on $[-1, 1]$, a function $f_{n,r}(x)$ having the following property:

1. It assumes its maximum and minimum values alternately at $n + r + 1$ points.
2. Its r -th derivative is equal in absolute value to one, has exactly n changes of sign, and $f_{n,r}(+1) = +1$.

For any integers $n \geq 0$, $r > 0$, and real β , $|\beta| > 1$, there exists, and is uniquely determined on the whole line, a function $f_{n,r,\beta}(x)$ satisfying the conditions:

1. It assumes its maximum and minimum values on the segment $[-1, 1]$ alternately at $n + r$ points.
2. Its r -th derivative is equal in absolute value to one and has there exactly n changes of sign.
3. $f_{n,r,\beta}^{(r)}(x) \equiv 1$ for $x \geq 1$; $f_{n,r,\beta}^{(r)}(x) \equiv f_{n,r,\beta}^{(r)}(-1)$ for $x \leq -1$;
 $f'_{n,r,\beta}(\beta) = 0$.

The functions $f_{n,r}(x)$ generalize the Chebyshev polynomials

$$f_{0,r}(x) = \frac{1}{r!} T_r(x) = \frac{1}{r! 2^{r-1}} \cos(r \arccos x).$$

The functions $f_{n,r,\beta}(x)$ generalize Zolotarev polynomials, coinciding with them also when $n = 0$. The functions $f_{n,r}(x)$ and $f_{n,r,\beta}(x)$ are solutions of a number of extremal problems connected with differential operators on the segment $[-1, 1]$. We indicate some of them.

The problem of widths of differentiable functions on a segment. Denote by W_r the class of functions $f(x)$, defined on $[-1, 1]$, for which

$$\sup_{x \in [-1, 1]} |f^{(r)}(x)| \leq 1. \quad (3)$$

Theorem 5*.

$$d_n(W_r) = d^n(W_r) = \begin{cases} \infty, & n \leq r - 1, \\ \|f_{n-r,r}\|, & n \geq r. \end{cases}$$

The space best approximating the set W_r is the space $L_{n,r}$ of dimension n , consisting of functions $f(x) = p(x) + \Phi(x)$, where $p(x)$ is a polynomial of degree

$\leq r - 2$, and the $r - 1$ derivatives of the function $\Phi(x)$ are piecewise constant on the intervals $\Delta_k = [\xi_k, \xi_{k+1}]$, where ξ_k are the break points of $f_{n-r,r}^{(r)}(x)$.

In order to approximate, with the necessary accuracy, a function $f(x) \in W_r$ by functions from L_{nr} , it is enough to interpolate it by functions from L_{nr} at the n points $\eta_1, \dots, \eta_n$, which are the zeros of $f_{n-r,r}(x)$. The space $L_{r+1,r}$ is not the only extremal one; for $n > r + 1$ the question has not been clarified.

In connection with Theorem 5 it seems of interest to us to point out the following fact.

Consider the class $\widetilde{W}_2$ of functions $f(x) \in [-1, 1]$ satisfying inequality (3) and periodic together with all derivatives up to order $r - 1$. The widths $d_{2n-1}(\widetilde{W}_r)$ were computed in (8). The extremal space turned out to be the subspace T_{2n-1} of trigonometric polynomials of degree $\leq n - 1$. It turns out that for no n is it

* In the case $r = 2$ this result was also obtained by R. L. Frum-Ketkov.

is the unique extremal subspace. Namely, the following holds.

Theorem 6. Apart from the space T_{2n-1} , the extremal space for the class $\widetilde{W}_r$ among $(2n)$ -dimensional spaces is the space $L_{2n,r}$ of functions $f(x)$ whose $(r - 1)$ -st derivative is piecewise constant on the intervals $\Delta_k = [\xi_k, \xi_{k+1}]$, where ξ_k are the zeros of the function $\sin nx$. In this case

$$d_{2n-1}(\widetilde{W}_r) = d_{2n}(\widetilde{W}_r).$$

For $r = 1$ this result was noted in (8).

The problem on inequalities for derivatives, i.e., the problem of computing

$$\max_{W_r \cap \alpha U} \|f^{(k)}(x)\| = F(k, \alpha); \quad \alpha > 0; \quad 0 < k < r$$

(U is the unit sphere in $C[-1, 1]$), or, in other words, the problem of finding

$$F(k, \alpha) = \max_{\substack{\|f\| \leq \alpha \\ f \in W_r}} \|f^{(k)}\|.$$

On the whole line this problem was solved by A. N. Kolmogorov (9). On a finite interval the following holds.

Theorem 7. For every α there exist unique $n_0(\alpha, k)$ and $\beta_0(\alpha, k)$ such that

$$F(k, \alpha) = \|f_{n_0, r, \beta_0}^{(k)}\|.$$

Theorem 7 generalizes V. A. Markov's inequality for polynomials.

The problem on inequalities for derivatives at a fixed point of the interval $[-1, 1]$.

Theorem 8. For any point $\xi \in [-1, 1]$ there exist $n_0(\xi, k)$, $\beta_0(\xi, k)$ such that

$$\max_{\substack{f \in W_r \\ \|f\| \leq \alpha}} |f^{(k)}(\xi)| = |f_{n_0, r, \beta_0}^{(k)}(\xi)|.$$

The problem of finding $\max_{|\xi| > 1} |f(\xi)| = G(\xi, \alpha)$ over all $f(x)$, for which $|f^{(r)}(x)| \leq 1$, $\|f\| = \max_{x \in [-1, 1]} |f(x)| \leq \alpha$.

Theorem 9. There exist $n_0(\xi, \alpha)$, $\beta_0(\xi, \alpha)$ such that $G(\xi, \alpha) = f_{n_0, r, \beta_0}(\xi)$.

Theorem 9 also generalizes the analogous property of Chebyshev polynomials.

I consider it my duty to note the great influence that conversations with A. A. Milyutin had on me, and to express to him my gratitude for this.

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