

ON THE THEORY OF OPERATORS OF THE FORM $\frac{d}{dt} - A$

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Abstract

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MATHEMATICS

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ON THE THEORY OF OPERATORS OF THE FORM $\frac{d}{dt} - A$

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We shall consider the equation

$$Lu \equiv \frac{du}{dt} - Au = f \quad (\text{L})$$

with boundary condition

$$\mu_1 u|_{t=0} - \mu_2 u|_{t=a} = 0, \quad ()$$

where, for $t \in [0, a]$, the functions $u(t), f(t)$ take values in a complex Hilbert space $\mathfrak{H}$; $\mu_1, \mu_2 : \mathfrak{H} \rightarrow \mathfrak{H}$ are bounded linear operators, and $A : \mathfrak{H} \rightarrow \mathfrak{H}$ is an unbounded normal operator commuting with d/dt . Equation (L) is a very popular object of investigation, but in this case it is usually assumed either that $\mu_2 = 0$ and A is the infinitesimal generator of a semigroup, or else one of the indicated assumptions is removed (for example, in ^(1,2)). In ⁽³⁾ it was shown that the use of conditions of the form (Γ) makes it possible to find “good” problems also for “pathological” operators L . In the present article the same question is considered from a more general point of view. We shall be interested in two types of conditions (Γ). In the first (problems L[Γ-I]) $\mu_2 = 1$ and $\mu_1 = \mu$ is a complex number. Conditions of this type make it possible to describe resolvable extensions ⁽³⁾ of the operator L in the case when the spectrum of the operator A does not fill the whole complex plane C , and include the “classical” problems. In conditions of the second type (problems L[Γ-II]) $\mu_1 = \mu^-(A)$, $\mu_2 = \mu^+(A)$ are special projection operators constructed from the operator A . These conditions make it possible to describe a resolvable extension of the operator L also for an operator A with spectrum filling the whole plane C . It should be noted that the results described pertain to a rather special class of normal operators A : to differential operators with constant coefficients on the n -dimensional torus.

Let us pass to the consideration of problems of the first type. If A is a numerical parameter, it is not difficult to write down an explicit formula giving the solution of problem L[Γ-I]. Introduce the notation:

$$S_t \equiv S(A, t) = e^{tA}; \quad \int_{\alpha}^{\beta} e^{(t-\tau)A} f(\tau) d\tau = I_{\alpha}^{\beta}(t, A)f.$$

Then the desired formula (under the assumption that $\mu - S_a \neq 0$) is written in the form:

$$u(t) = L^{-1}f(t) = (\mu I_0^t(t, A) + I_t^a(t, A)S_a)(\mu - S_a)^{-1}f. \quad (1)$$

If now A is a normal operator (i.e. $A = R + iQ$, where R, Q are commuting self-adjoint operators), formula (1) still retains its meaning under the assumption that μ is not a point of the discrete spectrum of the operator S_a .

Let T be an n -dimensional torus obtained by identifying opposite faces of a cube with edges of length 2π , referred to coordinates $(x_1, \dots, x_n)$. On T are defined a linear manifold P of sufficiently smooth functions and the complex Hilbert space H_x of functions with summable-

square with the usual scalar product and norm. If $A(s)$ is a polynomial with constant complex coefficients

$$A(s) = \sum_{|\alpha| \leq m} A_{\alpha} s^{\alpha}, \quad s^{\alpha} = s_1^{\alpha_1} \dots s_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n,$$

then $\dot{A} \equiv \dot{A}(-iD)$ is the corresponding differential operator, defined on P , and $\dot{A}(-iD)e^{is \cdot x} = A(s)e^{is \cdot x}$. By the operator $A : H_x \rightarrow H_x$ we mean the closure of $\dot{A}$ in H_x .

The subsequent exposition contains a number of more or less obvious propositions, the proofs of which are omitted.

Proposition 1. The operator A is normal, i.e. $AA^* = A^*A$.

Denote by $\mathfrak{S}$ the set of integer vectors $(s_1, \dots, s_n)$, where each s_i independently ranges over the set of values $0, \pm 1, \pm 2, \dots$, and by $A(\mathfrak{S})$ the set of values of the polynomial $A(s)$ for $s \in \mathfrak{S}$.

Proposition 2. The spectrum σA of the operator A consists of the closure in C of the set $A(\mathfrak{S})$.

In considerations on T , the role of the operational calculus that gives meaning to formula (1) is played by the use of the classical Fourier series.

Let now $H = H_x \otimes H_t$, where H_t is the Hilbert space $\mathcal{L}_2$ of functions defined on the interval $[0, a]$. By the operator $L : H \rightarrow H$ we shall mean the closure

in H of the operator of the left-hand side of equation (L), defined on smooth functions satisfying the conditions (Γ). Or: a function $u \in H$ belongs to $\mathfrak{D}_L$ if there exist sequences $\{u_i\}$ of smooth functions satisfying the conditions (Γ), and a function $f \in H$, such that $|u - u_i, H| \rightarrow 0$, $|Lu_i - f, H| \rightarrow 0$ as $i \rightarrow \infty$. The corresponding solution of problem L- Γ will be called a strong solution, or simply a solution. The operator L is **regular** if L^{-1} exists, is bounded, and is defined on all of H . In what follows, unless otherwise stated, the notation $\mu = \infty$ is used. It is assumed that in this case the problem in question is that for which $\mu_1 = 1$, $\mu_2 = 0$.

The set of values μ for which the operator L is not regular will be called the μ -**spectrum** of the problem L Γ and denoted by $\Sigma(A, a)$. By σA , $\sigma S(A, a)$, $\sigma L(A, a, \mu)$ we denote the spectra of the operators A , S_a , L . The investigation of problem L Γ is the study of various relations and dependencies among the spectra listed.

Theorem 1. *The equality*

$$\Sigma(A, a) = \sigma S(A, a)$$

holds.

The proof is quite elementary, but somewhat long. We give the main steps.

Proposition 3. The spectrum $\sigma S(A, a)$ is the closure in C of the set

$$\exp[aA(\mathfrak{S})],$$

which forms the discrete spectrum $\sigma_d S(A, a)$ of the operator S_a .

Denote by $\Sigma_d(A, a)$ the set of points μ of the plane C such that for $\mu \in \Sigma_d(A, a)$ the operator L^{-1} does not exist.

Proposition 4. The equality

$$\Sigma_d(A, a) = \sigma_d S(A, a)$$

is valid.

Lemma 1. *If $\mu \notin \sigma S(A, a)$, then the operator L is regular.*

The proof is based on the validity, in this case, of the energy inequality

$$|u, H| \leq c |Lu, H|, \quad ()$$

established by the methods used in ⁽³⁾.

Lemma 2. *If*

$$\mu \in \sigma_c S(A, a) \equiv \sigma S(A, a) \setminus \sigma_d S(A, a),$$

then the operator L^{-1} exists, is defined on a set dense in H , and is unbounded.

The existence of the operator L^{-1} is a trivial consequence of the uniqueness of the Fourier coefficients; unboundedness is established by considering a sequence $\{f^k\}_1^\infty$ of right-hand sides

$$f^k = \exp(is^k \cdot x),$$

where the sequence $\{s^k\}_1^\infty$ of vectors from $\mathfrak{S}$ is such that

$$|\mu - \exp[aA(s^k)]| \rightarrow 0$$

as $k \rightarrow \infty$. From Propositions 3 and 4 and Lemmas 1, 2, Theorem 1 follows.

In considering $\sigma L(A, a, \mu)$ it is convenient to introduce the notation $A_\lambda = \lambda E + A$.

Proposition 5. For given A, a, μ , a point $\lambda \in \sigma_d L(A, a, \mu)$ (or $\lambda \in \sigma_c L(A, a, \mu)$) if and only if $\mu \in \sigma_d S(A_\lambda, a)$ (or $\mu \in \sigma_c S(A_\lambda, a)$).

The following propositions clarify the special position of the Cauchy problem among the LG-I problems.

Proposition 6. If the point 0 (the point ∞) belongs to $\sigma S(A, a)$, then $\sigma L(A, a, 0)$ (respectively $\sigma L(A, a, \infty)$) fills the whole plane C . If $\mu \neq 0, \infty$, then $\sigma L(A, a, \mu)$ fills the whole plane if and only if the spectrum $\sigma S(A, a)$ fills the whole plane.

Proposition 7. The points $0, \infty$ cannot belong to $\Sigma_d(A, a)$.

Thus, in the compact case under consideration (the manifold $T \times [0, a]$ is compact), the uniqueness theory for the Cauchy problem is trivial. It follows from Proposition 6 that, if $\sigma S(A, a)$ does not fill C , one can always choose μ so that the LG-I problem will determine a solvable extension of the operator L .

Lemma 3. For $n < 3$ the spectrum σA cannot fill the whole plane C ; for $n \geq 3$ there exist operators A whose spectrum fills the whole plane.

The verification of the first part of the assertion is elementary. The second part is established by considering the operator for which

$$A(s) = s_1 + \alpha s_2 + i(s_3 + \beta s_2^2),$$

where α, β are irrational. Indeed, the fractional parts of the pairs

$$\{\alpha y, \beta y^2\}, \quad y = 0, \pm 1, \pm 2, \dots$$

are uniformly distributed in the unit square ⁽⁴⁾.

Thus, for $n \geq 3$ the LG-I problem does not allow one to describe a solvable extension of the operator L for arbitrary A .

Let us pass to the LG-II problem. Let the operator A be fixed and

$$A(s) = r(s) + iq(s),$$

where r, q are real polynomials. We divide the set $\mathfrak{S}$ into two subsets $\mathfrak{S}^-$, $\mathfrak{S}^+$, putting $s \in \mathfrak{S}^-$ if $r(s) \leq 0$ and $s \in \mathfrak{S}^+$ if $r(s) > 0$. The partition of $\mathfrak{S}$ induces a decomposition of H_x into the sum of orthogonal subspaces

$$H_x = H_x^- \oplus H_x^+,$$

where H_x^- (H_x^+) is the closed linear span of the set of vectors of the form $\exp(is \cdot x)$, $s \in \mathfrak{S}^-$ ($s \in \mathfrak{S}^+$). Let μ^-, μ^+ be the projection operators in H_x^-, H_x^+ , respectively.

Theorem 2. *For any operator A with constant coefficients on T , the LG-II problem in which $\mu_1 = \mu^-$, $\mu_2 = \mu^+$, determines a solvable extension of the operator L .*

For the proof it is enough to note that the coefficients of the expansion

$$u(x, t) = \sum_{\mathfrak{S}} u_s(t) e^{is \cdot x}$$

of an arbitrary smooth solution of the described problem will be determined from the equations

$$\begin{aligned} u_s(t) &= \int_0^t e^{(t-\tau)A(s)} f_s(\tau) d\tau, & s \in \mathfrak{S}^-, \\ u_s(t) &= - \int_t^a e^{(t-\tau)A(s)} f_s(\tau) d\tau, & s \in \mathfrak{S}^+, \end{aligned}$$

where $f_s(\tau)$ are the coefficients of the corresponding expansion of the right-hand side. An estimate of the form

$$|u_s(t)|^2 \leq |f_s(t) H_t|^2 \int_0^t e^{2(t-\tau)r(s)} d\tau$$

in the case $s \in \mathfrak{S}^-$, and an analogous estimate for the case $s \in \mathfrak{S}^+$, make it possible (by virtue of the corresponding sign of $r(s)$) to establish the energy inequality-

property (Φ) , whence the existence and uniqueness theorem for the solution for every $f \in H$ follows from (3) in the usual way.

The choice of the operators μ_1, μ_2 in problem LG-II, when describing the solvable extension, can of course be varied within fairly broad limits, but the use in the considerations of subspaces of the type $\mathfrak{S}^-, \mathfrak{S}^+$ (for $\sigma A = C$) seems to be dictated by the essence of the matter.

We note in conclusion that the approach described to the study of equation (L) induces a certain special classification of differential operators A with constant coefficients and of boundary-value problems LG-I according to the spectral properties of the operator S_a . The methods used evidently admit generalizations also to equations more general than (L).

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