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Abstract

Full Text

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DECOMPOSITION OF A REPRESENTATION OF A REDUCTIVE LIE ALGEBRA INTO REPRESENTATIONS OF ITS REGULAR REDUCTIVE SUBALGEBRAS OF MAXIMAL RANK

(Presented by Academician A. N. Kolmogorov on 25 XII 1964)

MATHEMATICS

1. Introduction.

Let g be a complex reductive algebra of rank l , g' its complex reductive regular subalgebra of the same rank, h a Cartan subalgebra of g and g' . Let Δ_+ be the set of positive roots φ_k of the algebra g ($k = 1, 2, \dots, (r-l)/2$; r is the order of the algebra g), and $\Delta'_+ \subset \Delta_+$ the set of positive roots φ_{k_i} of the subalgebra g' ($i = 1, 2, \dots, (r'-l')/2$; r' is the order of the subalgebra g'). Let, by definition, $\mu \in I$ if

$$2(\mu, \varphi_k)/(\varphi_k, \varphi_k) = \text{an integer for all } \varphi_k \in \Delta_+. \quad (1)$$

Similarly, let $\mu' \in I'$ if

$$2(\mu', \varphi_{k_i})/(\varphi_{k_i}, \varphi_{k_i}) = \text{an integer for all } \varphi_{k_i} \in \Delta'_+. \quad (2)$$

Since $\Delta'_+ \subset \Delta_+$, condition (2) is weaker than (1), i.e. $I \subset I'$. Recall that the set $I(I')$ is the set of all weights of all representations of the algebra $g(g')$. Let $\pi_\lambda(\pi'_{\lambda'})$ be an irreducible representation of the algebra g (of the subalgebra g') with highest weight λ (λ') in a finite-dimensional vector space $V_\lambda(V'_{\lambda'})$, and let $m_\lambda(m'_{\lambda'})$ be a function on $I(I')$ assigning to each vector $\nu \in I$ ($\nu' \in I'$) the multiplicity $m_\lambda(\nu)$ ($m'_{\lambda'}(\nu')$) of its occurrence as a weight in the representation $\pi_\lambda(\pi'_{\lambda'})$.

The aim of the present paper is to obtain a formula showing with what multiplicity $\rho_\lambda(\lambda')$ the irreducible representation $\pi'_{\lambda'}$ of the subalgebra g' occurs in the irreducible representation π_λ of the algebra g .

2. Formula for the multiplicity of an irreducible representation of the subalgebra.

We shall rely on two well-known general theorems of character theory:

$$\chi_\lambda(x) = \sum_{\nu \in I} m_\lambda(\nu) \exp(\nu, x); \quad (3)$$

$$\chi_\lambda(x) = \sum_{\lambda' \in I'_D} \rho_\lambda(\lambda') \chi'_{\lambda'}(x), \quad (4)$$

where (ν, x) is the Killing-Cartan bilinear form on g ; $x \in h$; $\chi_\lambda(x)$, $\chi'_{\lambda'}(x)$ are the characters of the representations $\pi_\lambda, \pi'_{\lambda'}$, respectively, and $\overline{I'_D}$ is the set of all highest weights of $\pi'_{\lambda'}$ (D' is a certain once and for all fixed Weyl chamber of the subalgebra g').**

* By definition, a subalgebra g is called a **reductive regular subalgebra of rank l** if it is the direct sum of a regular semisimple subalgebra of rank l' and a commutative algebra of rank $l - l'$. (A semisimple subalgebra g'' of the algebra g is called **regular** if its root system is part of the root system of the algebra g .)

** For more details on the definition of I'_D , see ⁽¹⁾. We note that by the Weyl group of the reductive algebra G one means the Weyl group of its semisimple part G' , extended identically to the Abelian complement to the Cartan subalgebra h' of the algebra G' .

Substituting (3) into (4), we have:

$$\sum_{\nu \in I} m_\lambda(\nu) \exp(\nu, x) = \sum_{\lambda' \in D'} \rho_\lambda(\lambda') \sum_{\nu' \in I'} m'_{\lambda'}(\nu') \exp(\nu', x). \quad (5)$$

Introduce the function $\delta_{\nu I}$, equal to one for $\nu \in I$ and equal to zero for $\nu \notin I$. Then, rewriting (5) in the form

$$\sum_{\nu \in I'} m_\lambda(\nu) \delta_{\nu I} \exp(\nu, x) = \sum_{\nu' \in I'} \sum_{\lambda' \in I'_D} \rho_\lambda(\lambda') m'_{\lambda'}(\nu') \exp(\nu', x) \quad (6)$$

and equating the coefficients of equal exponents, we obtain, for $\nu \in I'$,

$$m_\lambda(\nu) \delta_{\nu I} = \sum_{\lambda' \in I'_D} \rho_\lambda(\lambda') m'_{\lambda'}(\nu). \quad (7)$$

Our aim is to find an explicit form of $\rho_\lambda(\lambda')$. To this end we shall use Lemma 2.4 of [2], which can be formulated as follows.

Let $P(\mu)$ be the function of partitioning the vector μ into a sum of positive roots $\varphi_1, \varphi_2, \dots, \varphi_{k-1}, \varphi_k, \varphi_{k+1}, \dots, \varphi_n$, $n = (r - l)/2$. Then the first-order difference function

$$P^{(\varphi_k)}(\mu) = P(\mu) - P(\mu - \varphi_k) \quad (8)$$

is the function of partitioning the vector μ into a sum of positive roots $\varphi_1, \varphi_2, \dots, \varphi_{k-1}, \varphi_{k+1}, \dots, \varphi_n$. We note that the multiplicity of the weight ν of the representation π_λ of the algebra g is expressed in terms of $P(\mu)$ as follows:

$$m_\lambda(\nu) = \sum_{\sigma \in W} \text{sgn } \sigma P[\sigma(\beta + \lambda) - (\beta + \nu)], \quad (9)$$

which is a trivial generalization of Kostant's formula [1, 3] to reductive algebras. Here W is the Weyl group of the algebra g and

$$\beta = \frac{1}{2} \sum_1^n \varphi_k.$$

It follows directly from Lemma 2.4 of [2] that the difference function of order $n' = (r' - l')/2$

$$P^{(\Delta'_+)}(\mu) \equiv P^{(\varphi_{k_1}, \varphi_{k_2}, \dots, \varphi_{k_{n'}})} = \sum_{j_1=0}^1 \dots \sum_{j_{n'}=0}^1 (-1)^{\sum_1^{n'} j_i} P(\mu - j_i \varphi_{k_i}) \quad (10)$$

is the function of partitioning the vector μ into a sum of positive roots of the algebra g that do not belong to the subalgebra g' . We also note that the difference function of order n

$$P^{(\Delta_+)}(\mu) \equiv P^{(\varphi_1, \varphi_2, \dots, \varphi_n)}(\mu) = \delta_{\mu 0}, \quad (11)$$

since, by Lemma 2.4, $P^{(\Delta_+)}(\mu)$ is the function of partitioning the vector μ into a sum of zero roots.

Form the sum

$$m_\lambda^{(\Delta'_+)}(\nu) \equiv \sum_{j_1=0}^1 \dots \sum_{j_{n'}=0}^1 (-1)^{\sum_1^{n'} j_i} m_\lambda(\nu + j_i \varphi_{k_i}). \quad (12)$$

Substituting equality (7) into (12) and using formula (9) and definition (10), we obtain

$$\delta_{\nu I} \sum_{\sigma \in W} \text{sgn } \sigma P^{(\Delta'_+)}[\sigma(\beta + \lambda) - (\beta + \nu)] = \sum_{\lambda' \in I_{D'}} \rho_{\lambda}(\lambda') \sum_{\sigma' \in W'} \text{sgn } \sigma' P^{(\Delta'_+)}[\sigma'(\beta' + \lambda') - (\beta' + \nu)], \quad (13)$$

where $\nu \in I'$ and $P^{(\Delta'_+)}(\mu)$ is composed according to the prescription (10) from the functions of decomposition $P'(\mu)$ of the vector μ into the sum of positive roots φ_{k_i} of the subalgebra g' ; W' is the Weyl group of the subalgebra g' , and

$$\beta' = \frac{1}{2} \sum_{i=1}^{n'} \varphi_{k_i}.$$

Taking (11) into account, we have $P^{(\Delta'_+)}(\mu) = \delta_{\mu 0}$, i.e. (13) can be rewritten in the form

$$\delta_{\nu I} \sum_{\sigma \in W} \text{sgn } \sigma P^{(\Delta'_+)}[\sigma(\beta + \lambda) - (\beta + \nu)] = \sum_{\sigma' \in W'} \text{sgn } \sigma' \rho_{\lambda}[\sigma'^{-1}(\beta' + \nu) - \beta'], \quad (14)$$

where $\nu \in I'$, and the summation on the right-hand side of (14) is taken over all σ' such that $\sigma'^{-1}(\beta' + \nu) - \beta' \in I'_{D'}$.

It is easy to see that there exists only one σ' satisfying this condition, namely the Weyl reflection which takes the region $I'_{D'} + \beta'$ of the set I' into the region I' containing the vector $\nu + \beta'$. Since $\rho_{\lambda}(\nu)$ is defined only for $\nu \in I'_{D'}$, σ' must correspond to the identity element of the Weyl group, i.e. (14) takes the form

$$\rho_{\lambda}(\nu) = \sum_{\sigma \in W} \text{sgn } \sigma P^{(\Delta'_+)}[\sigma(\beta + \lambda) - (\beta + \nu)] \quad (15)$$

for $\nu \in I \cap I_{D'} = I_{D'}$, and $\rho_{\lambda}(\nu) = 0$ for $\nu \in I'_{D'}$.

Thus we finally obtain:

The multiplicity $\rho_{\lambda}(\lambda')$ of the irreducible representation $\pi'_{\lambda'}$ of the subalgebra g' , contained in the irreducible representation π_{λ} of the algebra g , is computed by the formula

$$\rho_{\lambda}(\lambda') = \sum_{\sigma \in W} \text{sgn } \sigma P^{(\Delta'_+)}[\sigma(\beta + \lambda) - (\beta + \lambda')] \quad (16)$$

for $\lambda' \in I_{D'}$, and $\rho_{\lambda}(\lambda') = 0$ for $\lambda' \in I'_{D'}$; $P^{(\Delta'_+)}(\mu)$ is the function of decomposition of the vector μ into a sum of positive roots of the algebra g which do not belong to the subalgebra g' .

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