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Abstract

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PHYSICS

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ELECTRODYNAMICS IN RIEMANNIAN SPACE

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It is of considerable interest to investigate in detail the equations of electrodynamics in Riemannian space, taking into account the general curvature of the Friedmann models of our Universe (the Metagalaxy). The basic results in this direction were obtained at one time by A. Eddington, who noted that the speed of light in "curved space," in a locally inertial system, is somewhat less than the fundamental speed of light.

It now makes sense to analyze more thoroughly these results of Eddington, and also those of Synge, who, having carried out analogous calculations, took into account only the local curvature of space produced by the electromagnetic field itself, which is insignificant, and did not take into account the general curvature of space, which is more essential.

Maxwell's equations have the form ⁽¹⁾

$$F_{;k}^{ik} = \frac{\partial(\sqrt{-g} F^{ik})}{\sqrt{-g} \partial x^k} = \frac{4\pi}{c} j^i; \quad F^{ik} = g^{ia} g^{kb} F_{ab}; \quad (1)$$

$$\partial F_{ik} / \partial x^l + \partial F_{li} / \partial x^k + \partial F_{kl} / \partial x^i = 0; \quad (2)$$

$$F_{ab} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} = \begin{vmatrix} 0 & H_z & -H_y & E_x \\ -H_z & 0 & H_x & E_y \\ H_y & -H_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{vmatrix}. \quad (3)$$

We shall write the Lorentz condition in the form ^(2,3)

$$g^{ik} A_{i;k} = 0. \quad (4)$$

Let us introduce the d' Alembert operator

$$\square A_a = \tilde{\square} A_a = g^{ik} A_{a;ik}. \quad (5)$$

First of all, let us note that the relation holds

$$\tilde{\square} A_a - g^{ik} A_{i;ak} + \frac{4\pi}{c} j_a = 0. \quad (6)$$

Further we find

$$\left(\tilde{\square} \delta_i^k - R_i^k \right) A_k = -\frac{4\pi}{c} j_i. \quad (7)$$

For a free field, when $j_i = 0$, we shall have

$$\left(\tilde{\square} \delta_i^k - R_i^k \right) A_k = 0. \quad (8)$$

In the case of an external Schwarzschild field $R_i^k = 0$ and

$$\tilde{\square} A_i = 0. \quad (9)$$

In the case of the general Friedmann field (the field of the Metagalaxy), calculations by known relations in the proper reference system lead to the following results:

$$R = \frac{6}{a^2} \left(\beta_1 + \frac{\dot{a}^2}{c^2} + \frac{a\ddot{a}}{c^2} \right); \quad R_0^0 = \frac{3a\ddot{a}}{c^2 a^2}; \quad R_\alpha^\beta = \frac{2\delta_\alpha^\beta}{a^2} \left(\beta_1 + \frac{\dot{a}^2}{c^2} + \frac{a\ddot{a}}{2c^2} \right), \quad (10)$$

where $\beta_1 = 1; 0; -1$ for three types of spaces (spherical, quasi-Euclidean, hyperbolic); $\dot{a} = da/d\tau$; $\ddot{a} = d^2a/d\tau^2$; a is the curvature, τ is proper time.

Thus,

$$\left(\tilde{\square} - \frac{3a\ddot{a}}{c^2 a^2} \right) A_0 = 0; \quad \left[\tilde{\square} - \frac{2}{a^2} \left(\beta_1 + \frac{\dot{a}^2}{c^2} + \frac{a\ddot{a}}{2c^2} \right) \right] A_\alpha = 0. \quad (11)$$

Since in the usual Friedmann models $R_0^0 = R_1^1$, putting

$$\frac{2}{a^2} \left(\beta_1 + \frac{\dot{a}^2}{c^2} + \frac{a\ddot{a}}{2c^2} \right) = -\frac{3a\ddot{a}}{c^2 a^2} = \xi_0^2 = \frac{R}{2},$$

we find $(\tilde{\square} - \xi_0^2)A_\alpha = 0$, where $\xi_0 = \sqrt{R/2}$ can be interpreted as the rest mass of the photon; $(\square + \xi_0^2)A_0 = 0$.

Assuming that $\chi \sim a \approx c\tau$, we obtain $\ddot{a} = 0$; $R_0^0 = 0$; $R_1^1 = \frac{2}{a^2} \left(\beta_1 + \frac{1}{\chi_0^2} \right) = \frac{R}{3}$, where $\frac{1}{\chi_0} = \frac{\dot{a}}{c}$; $\xi_0^2 = \frac{2}{a^2} \left(\beta_1 + \frac{1}{\chi_0^2} \right)$ for $i = 1, 2, 3$, $\xi_0^2 = 0$ for $i = 0$.

Proceeding from the fact that we must obtain the same equation

$$(\tilde{\square} - \xi_0^2)A_i = 0 \quad (12)$$

for all components A_i , it is necessary to allow a polarization of space and the emergence of a current. In the comoving frame of reference we shall have (for $\chi \sim a$)

$$j_\alpha = 0; \quad -j_0 = \frac{c}{4\pi} \frac{RA_0}{3} = \frac{cA_0}{2\pi a^2} \left(\beta_1 + \frac{1}{\chi_0^2} \right) = -\frac{\delta c g_{00}}{\sqrt{-g}}, \quad (13)$$

whence the density of charges induced by the change of χ is determined by the expression

$$\delta = \delta_\chi = -\frac{A_0}{2\pi a^2} \left(\beta_1 + \frac{1}{\chi_0^2} \right) \frac{\sqrt{-g}}{g_{00}} = -\frac{e}{2\pi r_0^2 a} \left(\beta_1 + \frac{1}{\chi_0^2} \right) = \frac{\delta_0}{T_m} = \frac{\delta_0}{\sqrt{N}},$$

where $\delta_0 \approx e/r_0^3$; N is the number of baryons in the Metagalaxy; T_m is the world dimensionless time. An analogous result will also be obtained for $\chi = \text{const}$. We shall interpret in equation (12) the quantity ξ_0^{-1} as the de Broglie wavelength $\lambda = \hbar/m_0 c$.

Thus, putting $\xi_0 = \beta_0/a = 1/\lambda = \sqrt{R/3}$, where $\beta_0 = [2(\beta_1 + 1/\chi_0^2)]^{1/2}$, we arrive at the result that the rest mass of the photon in Friedmann space is determined by the expression

$$m_0 = \beta_0 \hbar / ac. \quad (14)$$

At the same time, putting $\beta_0 = 2$, $a = 3 \cdot 10^{28}$ cm, we find $m_0 = 10^{-66}$ g, $m_0 c^2 = 10^{-45}$ erg, which corresponds to the energy of gravitons.

In some cases the equations of a strong gravitational field have the form (as shown by V. D. Zakharov):

$$(\tilde{\square} - {}^{1/2}\sqrt{\bar{R}}) R_{lmrt} = 0, \quad (15)$$

where $\tilde{\square} R_{lmrt} = g^{ik} R_{lmrt;ik}$; $\bar{R}^2 = R_{abcd} R^{abcd}$. Since $\bar{R} \simeq R \simeq \text{const}/a^2$, the rest mass of gravitons is again determined as $\approx 10^{-66}$ g.

One can draw the fundamental conclusion that in a Riemannian space the spatial components of the rest mass of quanta of the electromagnetic and gravitational fields are different from zero. With increasing a they decrease, and as $a \rightarrow \infty$, when space becomes Euclidean, the rest mass of photons and gravitons tends to zero.

An analogous problem, carried out only for a quasi-Euclidean Friedmann model of the Universe with $\chi = \text{const}$, was also considered by H. Nariai and T. Kimura⁽⁴⁾; however, they started from the equation $(\square - k/6)A_i = 0$ (the exact equation has the form: $(\square - R/2)A_i = 0$).

For quantization of the wave equation (12) it is expedient to pass to a locally geodesic coordinate system. In doing so we shall assume the operator $\square$ to be orthogonal, and R_i^k to be a tensor having only orthogonal terms, which are functions of world time. Then equations (12) (putting $c = 1$ and assuming that at the present time $\hbar = 1$) can be represented in the form

$$(\square - m_0^2)A_i = 0, \quad (16)$$

where $m_0 = \xi_0$.

In the process of quantization we shall regard m_0 as constant quantities, or, at least, as weakly dependent on the world time T_m (just as $\hbar$ is). Equation (16) has a formal similarity to the equation for a vector field.

The commutation relations can also be obtained by analogy with the quantized vector field (5)

$$[A_l^*(x), A_n(y)] = \frac{1}{i} \left(g_{ln} - \frac{1}{m_0^2} \frac{\partial^2}{\partial x^l \partial x^n} \right) D(x-y), \quad (17)$$

where $D(x-y)$ is the Pauli–Jordan function:

$$D(x) = \frac{1}{(2\pi)^3 i} \int e^{ikx} \delta(k^2 + m_0^2) \theta(k^0) dk, \quad (18)$$

with $k^0 = \pm \sqrt{\bar{k}^2 + m_0^2}$, where $\bar{k}$ is the momentum, $k^2 = \bar{k}^2 - k^{02}$.

These relations immediately show that the infrared catastrophe is automatically eliminated, since the minimal energy corresponding to the rest mass m_0 determines the maximum wavelength, and this maximum possible length cannot exceed the size of the system (the Metagalaxy).

The ultraviolet catastrophe is also eliminated. Starting from the uncertainty equation $\hbar = m^* r^* c$ and taking the gravitational field into account, putting $r^* = Gm^*/c^2$, one can calculate

$$r^* = \sqrt{\hbar G/c^3} \approx 10^{-33} \text{ cm}; \quad (19)$$

$$m^* = \sqrt{\hbar c/G} \approx 10^{-5} \text{ g}; \quad E^* \approx \sqrt{\hbar c^5/G} \approx 10^{16} \text{ erg}; \quad (20)$$

here $E^* = m^* c^2$ is the maximum possible photon energy, and r^* is the gravitational radius corresponding to this energy.

Since relation (20) can be written in the form $Gm^{*2}/\hbar c = aGm^{*2}/e^2 = 1$, it is evident that at $r = r^*$, $m = m^*$ the energy of the photon gravitational field corresponds to the energy of strong interactions, which should lead not only to gravitational-electromagnetic transmutations, but also to transmutations associated with the possible formation of mesons and other heavier particles.

Lengths of order $r^* = 10^{-33} \text{ cm}$ and times of order $\tau^* = r^*/c = 10^{-43} \text{ sec}$ are characteristic for describing fluctuations of the gravitational field and, as is now evident, are the minimal possible length and time in quantum field theory; these quantities can also characterize the quantum properties of space-time.

It is easy to ascertain that the quantity r^{*3} indeed characterizes volume fluctuations in the gravitational field. The number of quanta of the gravitational field (gravitons) is

$$N_g = E_M/m_0 c^2 = GM_M^2/\hbar c \simeq 10^{120},$$

where $E_M \simeq GM_M^2/a$ is the total energy of the gravitational field of the Metagalaxy, and M_M is its mass.

Since the volume of the Metagalaxy $V_M \simeq a^3$, the volume per graviton is

$$V_g = V_M/N_g = \hbar a^2/M_M c \simeq r_0^3 \simeq 10^{-38} \text{ cm}^3,$$

since $GM_M/c^2 a \simeq 1$, then $m_p/M_M \simeq r_0^2/a^2$, where r_0 is the "radius of the nucleon," and m_p is its mass. The fluctuation of the volume is

$$\Delta V_g \simeq V_g / \sqrt{N_g} = (G\hbar/c^3)^{3/2},$$

whence

$$\Delta r_g \simeq \Delta V_g^{1/3} \simeq \sqrt{G\hbar/c^3} \simeq r^*.$$

In exactly the same way, it is easy to see along the way that the quantity r_0^3 characterizes the fluctuation of volume for a field of strong (or electromagnetic) interactions.

Ordinary quantum field theory does not take into account the interaction of fields with the gravitational field, which is its essential shortcoming. Taking the gravitational field into account at very small and very large distances, when the energy of the gravitational field is comparable with the energy of particles or with the energy of other fields, evidently makes it possible to eliminate these divergences. Here two points of view on the possibility, in principle, of eliminating divergences are as it were united: taking account of the gravitational field and quantization of space–time, since the latter is a consequence of taking account precisely of gravitational interactions.

We see that the rest mass of photons and the energy of gravitons change (decrease) with time as R_i^k changes, proportionally to T_m^{-3} , remaining at all times in equilibrium with the energy density of the system (the Metagalaxy): $\rho_0 \approx GM_n^2/a^4 \sim T_m^{-3}$.

It is entirely natural to assume that the rest mass of elementary particles should likewise be in limiting thermodynamic equilibrium with the energy of the Metagalaxy and, consequently, should decrease with time, since the pressure of the metagalactic field also decreases with time.

Since in this formalism, in the general case, we can interpret R_i^k as certain quantities proportional to the “inertial tensor” rest mass of photons (and, in general, of any particles), the field equation may also be interpreted as an equation for certain masses (energies) having a tensor character and entering organically into the energy of any particles and their aggregate. Thus the curvature of space can indeed be interpreted energetically.

In essence this approach is completely opposite to Einstein’s purely geometrical method, although formally they are expressed by identical relations.

Let us further note that if one starts from the Hamilton–Jacobi equation

$$g^{ik}S_iS_k + m^2c^2 = 0 \quad (S_i = \partial S/\partial x^i),$$

writing it in the form

$$g^{00}S_0^2 + 2g^{0\alpha}S_0S_\alpha + g^{\alpha\beta}S_\alpha S_\beta + m^2c^2 = 0,$$

then for the energy we obtain the expression:

$$\sqrt{g^{00}} \frac{E_0}{c} = -\sqrt{g^{00}} S_0 - \frac{g^{0\alpha}S_\alpha}{\sqrt{-g^{00}}} \pm \sqrt{I^2 + m^2c^2 + \frac{(g^{0\alpha}S_\alpha)^2}{-g^{00}}},$$

where $S_\alpha = I_\alpha$, $I^2 = g^{\alpha\beta} S_\alpha S_\beta$.

It follows from this that the gravitational field will interact differently with particles and antiparticles and will violate combined inversion. In this case the change in the interaction energy and in the mass of particles will be of order $T_m^{-1} \approx 10^{-20}$ (in chronometrically invariant quantities this principle will not be violated);

This could lead to a significant gravitational differentiation of matter and anti-matter in the Metagalaxy, since at small T_m the differences in the interactions could have been considerable.

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Note: Figure translations are in progress. See original paper for figures.

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