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Abstract

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MATHEMATICS

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ON THE DENSITY OF FINITE FUNCTIONS AND THE EXTENSION OF CLASSES OF DIFFERENTIABLE FUNCTIONS

(Presented by Academician S. L. Sobolev on 19 IV 1965)

Let A be a domain of n -dimensional Euclidean space. Let E_n be such that from $x \in A$ it follows that $x + y \in A$, if $y = (y_1, \dots, y_n)$, $y_i \geq 0$ ($i = 1, \dots, n$).

We shall consider the linear manifold of locally summable functions $f(x)$ having generalized derivatives

$$D_i^{k_i} f(x) = \frac{\partial^{k_i}}{\partial x_i^{k_i}} f(x) \quad (i = 1, \dots, n)$$

with finite seminorm

$$\|f\|_{L_{p,\theta}^{l(\mathbf{m})}(A)} = \sum_{i=1}^n \left\{ \int_0^\infty \|\Delta_i^{s_i}(t) D_i^{k_i} f\|_{L_p(A)}^\theta \frac{dt}{t^{1+\theta(l_i-k_i)}} \right\}^{1/\theta}, \quad (1)$$

where $\mathbf{l} = (l_1, \dots, l_n)$, $\mathbf{m} = (m_1, \dots, m_n)$, $m_i = s_i + k_i > l_i > k_i \geq 0$, $1 \leq p \leq \infty$, $1 \leq \theta < \infty$,

$$\Delta_i^s(t) \varphi(x) = \sum_{j=1}^s (-1)^{k-j} C_k^j \varphi(x_1, \dots, x_{i-1}, x_i + jt, x_{i+1}, \dots, x_n),$$

$$\|f\|_{L_{p,\infty}^{l(\mathbf{m})}(A)} = \sum_{i=1}^n \operatorname{ess\,sup}_{t>0} t^{-l_i} \|\Delta_i^{s_i}(t) D_i^{k_i} f\|_{L_p(A)}.$$

Declaring functions $f(x)$ and $g(x)$ equivalent if almost everywhere $f(x) - g(x) = P_m(x)$, where $P_m(x)$ is a polynomial of degrees $m_1 - 1, \dots, m_n - 1$, we obtain the linear normed space $\mathcal{L}_{p,\theta}^{l(\mathbf{m})}(A)$ with norm (1), which is the quotient space by the space of polynomials $P_m(x)$.

We give a representation of functions from $\mathcal{L}_{p,\theta}^{l(\mathbf{m})}(A)$.

Let $\xi(z) \in C^\infty(E_1)$, $\xi(z) = 0$ for $x \leq 0$ and $x \geq 1$,

$$\int_0^\infty \xi(z) dz = 1,$$

$$\omega_m(z) = \sum_{k=1}^m (-1)^{k+1} C_m^k \frac{1}{k} \xi\left(\frac{z}{k}\right).$$

Hence

$$\int_0^\infty \omega_m(z) \varphi(x + hz) dz - \varphi(x) = (-1)^{m+1} \int_0^\infty \xi(z) \Delta^m(hz) \varphi(x) dz.$$

Put, for $h_j > 0$ ($j = 1, \dots, n$),

$$f_{h_1, \dots, h_n}(x) = \int_0^\infty \dots \int_0^\infty \prod_{j=1}^n \frac{1}{h_j} \omega_{m_j}\left(\frac{z_j}{h_j}\right) f(x + z) dz,$$

$$\tilde{f}_{h_1, \dots, h_n}(x) = \int_0^\infty \dots \int_0^\infty \prod_{j=1}^n \frac{1}{h_j^2} \omega_{m_j}\left(\frac{z_j}{h_j}\right) \omega_{m_j}\left(\frac{y_j}{h_j}\right) f(x + z + y) dz dy, \quad (2)$$

$$\hat{f}_{h_1, \dots, h_n}(x) = 2f_{h_1, \dots, h_n}(x) - \tilde{f}_{h_1, \dots, h_n}(x).$$

As $\sum_1^n h_i \rightarrow 0$, $\hat{f}_{h_1, \dots, h_n}(x) \rightarrow f(x)$ in the sense of convergence in L_{loc} and almost everywhere.

Let $h_i = h^{\sigma_i}$, $\sigma_i > 0$. Then

$$\frac{\partial}{\partial h} [\hat{f}_{h^{\sigma_1}, \dots, h^{\sigma_n}}(x)] = - \sum_{i=1}^n \int_0^\infty \dots \int_0^\infty h^{-1 - \sum_{j=1}^n \sigma_j - \sigma_i} \chi_i\left(\frac{y_1}{h^{\sigma_1}}, \dots, \frac{y_n}{h^{\sigma_n}}\right) \times \xi\left(\frac{t}{h^{\sigma_i}}\right) \Delta_i^{m_i}(t) f(x+y) dt dy,$$

where the function $\chi_i(y_1, \dots, y_n) \in C^\infty(E_n)$ is concentrated in the parallelepiped $0 \leq y_k \leq 2m_k$ ($k = 1, \dots, n$). Integrating with respect to h , for almost all x , for any $H > 0$ we obtain the representation

$$f(x) = \hat{f}_{H^{\sigma_1}, \dots, H^{\sigma_n}}(x) + \int_0^H \sum_{i=1}^n \int_0^\infty \dots \int_0^\infty h^{-1 - \sum_{j=1}^n \sigma_j - \sigma_i} \chi_i\left(\frac{y_1}{h^{\sigma_1}}, \dots, \frac{y_n}{h^{\sigma_n}}\right) \times \xi\left(\frac{t}{h^{\sigma_i}}\right) \Delta_i^{m_i}(t) f(x+y) dt dy dh. \quad (3)$$

For functions $\psi_i(t, x)$, $(t, x) \in (0, \infty) \times A = A^+$, with finite norms

$$\|\psi_i\|_{L_{p,\theta}(A^+)} = \|\|\psi_i\|_{L_p(A)}\|_{L_\theta(0,\infty)}, \quad \sigma_i = 1/l_i,$$

put

$$g_{(\delta,H)}(x) = \int_\delta^H \sum_{i=1}^n \int_0^\infty \dots \int_0^\infty h^{-1-\sum_{j=1}^n \sigma_j - \sigma_i} \chi_i\left(\frac{y_1}{h^{\sigma_1}}, \dots, \frac{y_n}{h^{\sigma_n}}\right) \times \xi\left(\frac{t}{h^{\sigma_i}}\right) t^{1/\theta+l_i} \psi_i(t, x+y) dt dy dh. \quad (4)$$

With the aid of Hardy' s and Minkowski' s inequalities, after the corresponding changes of variables one can show that, for $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$,

$$\|g_{(\delta,H)}\|_{\mathcal{L}_{p,\theta}^{1(\mathbf{m})}(A)} \leq C \sum_{i=1}^n \int_0^1 u^{l_i} \left\{ \int_{u\delta_i}^{uH^{\sigma_i}} \|\psi_i\|_{L_p(A)}^\theta dt \right\}^{1/\theta} du \leq C \sum_{i=1}^n \|\psi_i\|_{L_{p,\theta}(A^+)}. \quad (5)$$

Corollary. Since

$$\sum_{i=1}^n \|t^{-1/\theta-l_i} \Delta_i^{m_i}(t) f(x)\|_{L_{p,\theta}(A^+)} \leq \|f\|_{\mathcal{L}_{p,\theta}^{1(\mathbf{m})}(A)},$$

it follows from (3), (4), (5) that, for $\sigma_i = 1/l_i$, $H \rightarrow 0$,

$$\|f - \hat{f}_{H^{\sigma_1}, \dots, H^{\sigma_n}}\|_{\mathcal{L}_{p,\theta}^{1(\mathbf{m})}(A)} \rightarrow 0,$$

in any case when $1 \leq \theta < \infty$, and for those functions from $\mathcal{L}_{p,\infty}^{1(\mathbf{m})}(A)$ for which

$$\|\Delta_i^{m_i}(t) f\|_{L_p(A)} = o(t^{l_i}) \quad \text{as } t \rightarrow 0 \quad (i = 1, \dots, n),$$

i.e. the density of the set $C^\infty(\bar{A})$ in these classes.

For $H = \infty$, the right-hand side of (4) is understood as an integral convergent in itself in $\mathcal{L}_{p,\theta}^{1(\mathbf{m})}(A)$ as $H \rightarrow \infty$, which holds, in any case, for $1 \leq \theta < \infty$, and for $\theta = \infty$ for those functions $\psi_i(t, x)$ for which

$$\|\psi_i\|_{L_p(A)} = o(1) \quad \text{as } t \rightarrow \infty.$$

Let

$$\varphi^{(i)}(t, x) = t^{-1/\theta-l_i+k_i} \Delta_i^{s_i}(t) D_i^{k_i} f(x).$$

As $\sum_1^n h_i \rightarrow \infty$

$$\|f_{h_1, \dots, h_n}\|_{L_{p, \theta}^{1(\mathbf{m})}(A)} = \sum_{i=1}^n \|\varphi_{h_1, \dots, h_n}^{(i)}\|_{L_{p, \theta}(A+)} \rightarrow 0, \quad \|\hat{f}_{h_1, \dots, h_n}\|_{L_{p, \theta}^{1(\mathbf{m})}(A)} \rightarrow 0 \quad (6)$$

at least for $1 < p \leq \infty^*$, $1 \leq \theta \leq \infty$, and under the following additional requirements:

a) for $p = \infty$,

$$\left\{ \int_{\alpha}^{\beta} |\varphi^{(i)}(t, x)|^{\theta} dt \right\}^{1/\theta} \rightarrow 0 \quad (x \in A, |x| \rightarrow \infty),$$

for which it is sufficient that

$$\sum_1^n |D_i^{k_i}(f - P_m(f; x))| \rightarrow 0 \quad (x \in A, |x| \rightarrow \infty);$$

b) for $\theta = \infty$,

$$\|\varphi^{(i)}(t, x)\|_{L_p(A)} \rightarrow 0 \quad (t \rightarrow 0, t \rightarrow \infty),$$

$$\operatorname{ess\,sup}_{\alpha < t < \beta} \sum_{i=1}^n \|\varphi^{(i)}(t, x)\|_{L_p(x \in A, |x| > R)} \rightarrow 0 \quad (R \rightarrow \infty)$$

for every $[\alpha, \beta] \subset (0, \infty)$. The last condition, for $1 < p < \infty$, is fulfilled if

$$\sum_{i=1}^n \|D_i^{k_i}(f - P_m(f; x))\|_{L_p(A)} < \infty.$$

In the proof one uses the approximation of the functions $\varphi^{(i)}(t, x)$ in $L_{p, \theta}(A+)$ by finite and bounded ones.

When (6) holds, from (3) we obtain the representation in $\mathcal{L}_{p, \theta}^{1(\mathbf{m})}(A)^{**}$

$$\begin{aligned} f(x) = & \int_0^\infty \sum_{i=1}^n \int_0^\infty \dots \int_0^\infty h^{-1-\sum_{j=1}^n \sigma_j - \sigma_i} \chi_i\left(\frac{y_1}{h^{\sigma_1}}, \dots, \frac{y_n}{h^{\sigma_n}}\right) \times \\ & \times \xi\left(\frac{t}{h^{\sigma_i}}\right) \Delta_i^{m_i}(t) f(x+y) dt dy dh. \end{aligned} \quad (7)$$

For finite $f(x) \in C^\infty(\bar{A})$, (3) and (7) hold everywhere in A and give representations analogous to the representations of V. P. Il' in.

Theorem 1. The functions $f(x)$ of the space $\mathcal{L}_{p, \theta}^{1(\mathbf{m})}(A)$, $1 < p \leq \infty$, $1 \leq \theta \leq \infty$ (for $p = \infty$ or $\theta = \infty$ satisfying the additional conditions a), b)) can be approximated with arbitrary accuracy by infinitely differentiable finite functions.

For the proof it is enough to observe that any such function $f(x)$ is approximated with arbitrary accuracy by the functions $(0 < \delta < H < \infty)$

$$f_{(\delta, H)}(x) = \int_{\delta}^H \sum_{i=1}^n \int_0^{\infty} \dots \int_0^{\infty} h^{-1-\sum_{j=1}^n \sigma_j - \sigma_i} \chi_i \left(\frac{y_1}{h^{\sigma_1}}, \dots, \frac{y_n}{h^{\sigma_n}} \right) \times \\ \times \xi \left(\frac{t}{h^{\sigma_i}} \right) \Delta_i^{m_i}(t) f(x+y) dt dy dh,$$

and the function $f_{(\delta, H)}(x)$ (for fixed $\delta > 0$, $H < \infty$) by functions $g_{(\delta, H)}(x)$ with finite (and infinitely differentiable) functions ψ_i with compact support in $A+$ (approximating $t^{-1/\theta-l_i} \Delta_i^{m_i}(t) f(x)$). But such functions $g_{(\delta, H)}(x) \in C^{\infty}(\bar{A})$ are finite, as was required to prove.

Theorem 2. The space $\mathcal{L}_{p, \theta}^{1(m)}(A)$, $1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$, is complete.

* The same also holds for $p = 1$ for a domain A which, for some i , contains entirely no straight line $x_k = x_k^0$ ($1 \leq k \leq n$, $k \neq i$) and $h_i \rightarrow \infty$.

** The right-hand side of (7), for every $R < \infty$, converges in $L_p(|x| < R, x \in A)$ to $f(x) - P(f, x)$.

The proof is carried out by projecting $\mathcal{L}_{p, \theta}^{1(m)}(A)$ onto the space of polynomials $P_m(x)$.

Remark. Let $\tilde{\mathcal{L}}_{p, \theta}^{1(m)}(A)$ denote the closure of finite functions $f(x) \in C^{\infty}(\bar{A})$ in the norm $\mathcal{L}_{p, \theta}^{1(m)}(A)$. Such spaces were studied in (2). For $1 < p < \infty$, $1 \leq \theta < \infty$, $\tilde{\mathcal{L}}_{p, \theta}^{1(m)}(A)$ coincides with $\mathcal{L}_{p, \theta}^{1(m)}(A)$. From estimate (5) it follows that, for $1 \leq p < \infty$, $1 \leq \theta < \infty$, $\tilde{\mathcal{L}}_{p, \theta}^{1(m)}(A)$ coincides with the functions $g_{(0, \infty)}(x)$ representable in the form (4) through functions

$$\psi_i(t, x), \quad \sum_{i=1}^n \|\psi_i\|_{L_{p, \theta}(A^+)} < \infty.$$

The question of the coincidence of $\tilde{\mathcal{L}}_{1, \theta}^{1(m)}(E_n)$ and $\mathcal{L}_{1, \theta}^{1(m)}(E_n)$ remains open.

Theorem 3. Let

$$s_i + k_i \geq m_i > l_i > k_i \geq 0 \quad (i = 1, \dots, n).$$

In the space $\mathcal{L}_{p, \theta}^{1(m)}(A)$, $1 < p < \infty$, $1 \leq \theta < \infty$, all norms of the form (1) are equivalent for different s_i, k_i , and on finite functions $f(x) \in C^{\infty}(\bar{A})$ ($1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$)—for different s_i, k_i, m_i .

The second part of the theorem is known (3). For the proof it is convenient to use representation (7) and the same estimates as in the derivation of (5).

Theorem 4. The spaces $\tilde{\mathcal{L}}_{p,\theta}^{1(m)}(A)$ ($1 \leq p \leq \infty$, $1 \leq \theta \leq \infty$) admit a linear bounded extension from A to E_n .

The proof is carried out by extending the functions

$$\varphi_i(t, x) = \Delta_i^{m_i}(t)f(x)$$

in (7) by zero for $x \in E_n \setminus A$ and using estimates (5). The idea of this method belongs to Calderon ⁽⁴⁾; for $l_1 = \dots = l_n$ an analogous result was obtained in ⁽⁵⁾; the possibility of applying the method in the present situation was also suggested by V. P. Il' in.

An analogous theorem is also valid for the spaces $B_{p,\theta}^l(A)$ (introduced in (6)) with the normalization

$$\|f\|_{B_{p,\theta}^l(A)} = \|f\|_{L_p(A)} + \|f\|_{\tilde{\mathcal{L}}_{p,\theta}^{1(m)}(A)}.$$

The possibility of extending $\tilde{\mathcal{L}}_{p,\theta}^{1(m)}(A)$ for

$$A = E_m \times (0, \infty)^{n-m}$$

was established by V. A. Solonnikov by the method of Whitney and Hestenes ⁽¹⁾.

Let, for a domain Ω , U_δ denote the intersection of the n -dimensional δ -neighborhood of the set U with Ω . Suppose that for Ω there exists a finite covering $\{U^{(k)}\}_{k=0}^K$ with the properties:

- 1) $\Omega \subset \bigcup_{k=0}^K U^k \subset \Omega$,
- 2) there exist cones

$$V_k\{(x, Q_k) > |x| \cos \gamma, |x| < r\}$$

such that, for $x \in U_{2\delta}^{(k)}$, $x + V_k \subset \Omega$, and for

$$x \in \partial\Omega \cap \bar{U}_{2\delta}^{(k)}$$

one has $x - V_k \subset E_n \setminus \Omega$. These conditions are satisfied, for example, by a bounded domain Ω with an $(n-1)$ -dimensional boundary $\partial\Omega$ locally satisfying the Lipschitz condition. Put

$$\|f\|_{B_{p,\theta}^l(\Omega)} = \sum_{|\alpha| \leq \bar{l}} \|D^\alpha f\|_{L_p(\Omega)} + \sum_{|\alpha| \leq \bar{l}} \left\{ \int_{E_n} \|\Delta^2(t) D^\alpha f\|_{L_p(\Omega)^{(2t)}}^\theta \frac{dt}{|t|^{m+\theta\lambda}} \right\}^{1/\theta},$$

where $\bar{l}$ is an integer; $0 < l - \bar{l} = \lambda \leq 1$; $\Omega^{(t)}$ is the largest domain for which

$$\bigcap_{0 \leq \rho \leq 1} (\Omega^{(t)} + \rho t) \subset \Omega.$$

For $\lambda < 1$, $\Delta^2(t)$, $\Omega^{(2t)}$ may be replaced by $\Delta(t)$, $\Omega^{(t)}$.

Theorem 5. *The spaces $B_{p,\theta}^l(\Omega)$ admit a linear bounded extension of functions from Ω to E_n . The spaces $B_{p,\theta}^l(A)$ and $B_{p,\theta}^l(A)$ coincide for $l = (l, \dots, l)$; their norms are equivalent.*

Correction note. After the paper had been submitted to the editorial office, it became known to us that the theorem on the extension of functions from the space $B_{p,\theta}^l$ had also been proved by V. P. Il' in.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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