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Abstract

Full Text

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ON THE MODULI OF CONVEXITY AND SMOOTHNESS OF BANACH SPACES

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Mathematics

The **modulus of convexity** and, respectively, the **modulus of smoothness** (see, for example, ⁽¹⁾) of a Banach space E are the functions

$$\delta_E(\omega) = \inf_{\substack{\|x\|=\|y\|=1 \\ \|x-y\|\geq\omega}} (1 - \|x+y\|/2) \quad (0 \leq \omega \leq 2)$$

and, respectively,

$$\alpha_E(\omega) = \frac{1}{2} \sup_{\|x\|=1, \|y\|=\omega} (\|x+y\| + \|x-y\| - 2) \quad (\omega \geq 0).$$

The space E is called **uniformly convex**, respectively **uniformly smooth**, if $\delta_E(\omega) > 0$, $0 < \omega \leq 2$, respectively, if

$$\lim_{\omega \rightarrow 0} \alpha_E(\omega)/\omega = 0.$$

As M. M. Day ⁽²⁾ showed, in order that E be uniformly smooth, it is necessary and sufficient that E^* be uniformly convex. J. Lindenstrauss ⁽¹⁾ gave this criterion an exact form, establishing the following duality relation:

$$\alpha_{E^*}(\omega) = \sup_{0 \leq \varepsilon \leq 2} (\omega\varepsilon/2 - \delta_E(\varepsilon)) \cdot (\omega > 0).$$

In the present note new definitions of the moduli of convexity and smoothness of a Banach space are introduced, leading to the same classes of uniformly convex and, respectively, uniformly smooth spaces as those indicated above, but for which it is possible to obtain simpler duality relations. As an application of certain properties of the moduli of convexity and smoothness, a generalization is obtained of a theorem of Krein–Krasnosel' skii–Mil' man ⁽³⁾ on the aperture of subspaces, and one criterion is established for the instability of the property of incompleteness of a sequence in a uniformly smooth space.

Our starting point will be the notion, introduced in ⁽³⁾, of the aperture of subspaces P and Q of a Banach space E :

$$\theta(P, Q) = \max \left\{ \sup_{x \in P, \|x\|=1} \rho(x, Q), \sup_{y \in Q, \|y\|=1} \rho(y, P) \right\},$$

where $\rho(z, R)$ denotes the distance of the element z from the subspace R .

Definition. By the **modulus of convexity** of the Banach space E we shall mean the function

$$\beta_E(\omega) = \inf_{\substack{\|x\|=\|y\|=1 \\ \theta(L_x, L_y) \geq \omega}} \max_{0 \leq t \leq 1} (1 - \|tx + (1-t)y\|), \quad 0 \leq \omega \leq 1,$$

where L_z denotes the one-dimensional subspace in E generated by the element z . By the **modulus of smoothness** of the space E we shall mean the function

$$\gamma_E(\omega) = \inf_{\substack{x \in E, \|x\|=1 \\ \theta(P_1, P_2) \geq \omega}} \max\{1 - \rho(x, P_1), 1 - \rho(x, P_2)\}, \quad 0 \leq \omega \leq 1,$$

where P_1, P_2 are hyperplanes in E . Obviously, $\beta_E(\omega)$ and $\gamma_E(\omega)$ are continuous monotonically nondecreasing functions for $0 \leq \omega \leq 1$.

In the subsequent exposition we shall assume the space E to be real; however, all the facts presented are also true in the case of complex spaces.

Theorem 1. For every Banach space E ,

$$\delta_E(\omega) \leq \beta_E(\omega) \leq 2\delta(2\omega), \quad 0 \leq \omega \leq 1. \quad (1)$$

Proof. Obviously, it is enough to prove (1) for the case $\dim E = 2$; moreover, we may regard E as a Euclidean plane, renormed by means of the Minkowski metric with respect to some convex centrally symmetric curve $\mathcal{T}$ —the unit sphere in E . For the given ω , $0 \leq \omega \leq 1$, let $x \in E$ and $y \in E$, $\|x\| = \|y\| = 1$, be such that $\|x - y\| = 2\omega$ and $1 - \|(x + y)/2\| = \delta_E(2\omega)$.

Identify x and y with the vectors, respectively, $\overline{OA}$ and $\overline{OB}$, $A \in \mathcal{T}$, $B \in \mathcal{T}$ (O is the center of $\mathcal{T}$); let $\overline{OC} = \frac{1}{2}(\overline{OA} + \overline{OB})$, and let D be the point of intersection of the extension of OC with $\mathcal{T}$ (D and O lie on different sides of AB); draw $B\mathcal{E} \parallel CD$, where $\mathcal{E}$ lies on the extension of AD . We shall show that for $0 \leq t \leq 1$

$$1 - \|tx + (1-t)y\| \leq 2(1 - \|x + y\|/2).$$

Let $tx + (1-t)y = \overline{OF}$; let M be the point of intersection of the extension of OF with $\mathcal{T}$; draw $MP \parallel AB$ and $EN \parallel AB$ to their intersections with the extension of OD at the points P and, respectively, N . We have

$$\begin{aligned} 1 - \|tx + (1-t)y\| &= 1 - \|\overline{OF}\| = \|\overline{MF}\|/\|\overline{OM}\| = \|\overline{CP}\|/\|\overline{OP}\| \leq \\ &\leq \|\overline{CN}\|/\|\overline{NO}\| \leq \|\overline{CN}\|/\|\overline{DO}\| = 2\|\overline{CD}\|/\|\overline{DO}\| = 2(1 - \|x + y\|/2). \end{aligned}$$

We may suppose (interchanging x and y , if necessary) that there is a $z \in L_y$ such that $\|x - z\| = \theta(L_x, L_y)$ and $z = \alpha y$ for some α . Suppose first that $\alpha \geq 0$; then

$$\|y - z\| = 1 - \|z\| \leq 1 - (\|x\| - \|z - x\|) = \|z - x\|.$$

Therefore

$$2\omega = \|x - y\| \leq \|z - x\| + \|y - z\| \leq 2\|x - z\| = 2\theta(L_x, L_y),$$

whence $\theta(L_x, L_y) \geq \omega$. Thus we have

$$\beta_E(\omega) \leq \max_{0 \leq t \leq 1} (1 - \|tx + (1-t)y\|) \leq 2(1 - \|x + y\|/2) = 2\delta_E(2\omega).$$

If, however, $\alpha < 0$, then, continuously moving the points A and B along $\mathcal{T}$, we find points $A' \in \mathcal{T}$, $B' \in \mathcal{T}$ such that $A'B' \parallel AB$, $A'B'$ is farther from O than AB , and if we denote $OA' = x'$, $OB' = y'$, then for some $\alpha' \geq 0$ $\theta(L_{x'}, L_{y'}) = \|x' - \alpha'y'\| = \omega$ (the possible case $\|y' - \alpha'x'\| = \omega$ is treated analogously); therefore we have

$$\begin{aligned} \delta_E(2\omega) &\geq \frac{1}{2} \max_{0 \leq t \leq 1} (1 - \|tx + (1-t)y\|) \geq \\ &\geq \frac{1}{2} \max_{0 \leq t \leq 1} (1 - \|tx' + (1-t)y'\|) \geq \frac{1}{2} \beta_E(\omega). \end{aligned}$$

Thus, in both cases, $\beta_E(\omega) \leq 2\delta_E(2\omega)$, $0 \leq \omega \leq 1$. The inequality $\delta(\omega) \leq \beta(\omega)$, $0 \leq \omega \leq 1$, follows directly from the definitions of $\delta_E(\omega)$ and $\beta_E(\omega)$, if one takes into account that for $\|x\| = \|y\| = 1$ one has $\|x - y\| \geq \theta(L_x, L_y)$. The theorem is proved.

Theorem 2. For every Banach space E ,

$$\gamma_E(\omega) = \beta_{E^*}(\omega), \quad \beta_E(\omega) = \gamma_{E^*}(\omega), \quad 0 \leq \omega \leq 1.$$

Proof. For the given $\varepsilon > 0$ and ω , $0 \leq \omega \leq 1$, there exist $f_1 \in E^*$, $f_2 \in E^*$, $\|f_1\| = \|f_2\| = 1$, such that $\theta(L_{f_1}, L_{f_2}) \geq \omega$ and

$$\max_{0 \leq t \leq 1} (1 - \|tf_1 + (1-t)f_2\|) \leq \beta_{E^*}(\omega) + \varepsilon. \quad (2)$$

Consider the family of functionals

$$\Phi_t = \frac{tf_1 + (1-t)f_2}{\|tf_1 + (1-t)f_2\|}, \quad 0 \leq t \leq 1.$$

There exist elements $x \in E$, $y \in E$ such that

$$\|x\| = \|y\| = 1, \quad f_1(x) > 1 - \varepsilon, \quad f_2(y) > 1 - \varepsilon. \quad (3)$$

Obviously, there is a set $\{t_i\}_{i=0}^n$, $0 = t_0 < t_1 < \dots < t_n = 1$, such that

$$\|\Phi_{t_{i+1}} - \Phi_{t_i}\| < \varepsilon/(1 + \varepsilon). \quad (4)$$

Consider the segmental surfaces σ_i : $z \in \sigma_i$ means that $\|z\| = 1$, $\Phi_i(z) > 1/(1 + \varepsilon)$, $i = 0, 1, \dots, n$. Obviously, σ_i is connected; by (4), $\sigma_i \cap \sigma_{i+1}$ is nonempty ($i = 0, 1, \dots, n-1$); hence $\sigma = \bigcup_{i=0}^n \sigma_i$ is connected. The functional $F = f_1 - f_2$ is a continuous function of $z \in \sigma$; by (3), $F(x) > -\varepsilon$, $F(y) < \varepsilon$, and, consequently, there is $z_0 \in \sigma$ such that $|F(z_0)| < \varepsilon$, i.e.

$$|f_1(z_0) - f_2(z_0)| < \varepsilon, \quad \|z_0\| = 1. \quad (5)$$

Moreover, $z_0 \in \sigma_{j_0}$ for some j_0 , $0 \leq j_0 \leq n$, i.e. $\Phi_{t_{j_0}}(z_0) > 1/(1 + \varepsilon)$. This means that

$$|t_{j_0}f_1(z_0) + (1 - t_{j_0})f_2(z_0)| \geq (1 + \varepsilon)^{-1} \|t_{j_0}f_1 + (1 - t_{j_0})f_2\|. \quad (6)$$

Let P_1 and P_2 be hyperplanes in E annihilating f_1 and f_2 , respectively. We have

$$|f_1(z_0)| = \rho(z_0, P_1), \quad |f_2(z_0)| = \rho(z_0, P_2),$$

$$\max\{1 - \rho(z_0, P_1), 1 - \rho(z_0, P_2)\} \geq \gamma_E(\omega).$$

Therefore, by (5): $|f_1(z_0)| < 1 - \gamma_E(\omega) + \varepsilon$, $|f_2(z_0)| < 1 - \gamma_E(\omega) + \varepsilon$. But then

$$|t_{j_0}f_1(z_0) + (1 - t_{j_0})f_2(z_0)| < 1 - \gamma_E(\omega) + \varepsilon$$

and from (6) we obtain

$$\begin{aligned}\|t_{j_0}f_1 + (1 - t_{j_0})f_2\| &< (1 + \varepsilon)(1 - \gamma_E(\omega) + \varepsilon) = \\ &= 1 - \gamma_E(\omega) + 2\varepsilon - \varepsilon\gamma_E(\omega) + \varepsilon^2,\end{aligned}$$

whence, taking into account (2) and the fact that $\theta(P_1, P_2) = \theta(L_{f_1}, L_{f_2})$ ⁽³⁾,

$$\begin{aligned}\beta_{E^*}(\omega) + \varepsilon &\geq \max_{0 \leq t \leq 1} (1 - \|tf_1 + (1 - t)f_2\|) > 1 - \|t_{j_0}f_1 + (1 - t_{j_0})f_2\| > \\ &> \gamma_E(\omega) - 2\varepsilon + \varepsilon\gamma_E(\omega) - \varepsilon^2,\end{aligned}$$

and, since $\varepsilon > 0$ is arbitrary, $\beta_{E^*}(\omega) \geq \gamma_E(\omega)$. We prove the reverse inequality. For given $\varepsilon > 0$ and ω , $0 \leq \omega \leq 1$, there exist $x \in E$, $\|x\| = 1$ and hyperplanes P_1 and P_2 in E , $\theta(P_1, P_2) \geq \omega$, such that

$$\max\{1 - \rho(x, P_1), 1 - \rho(x, P_2)\} \leq \gamma_E(\omega) + \varepsilon.$$

Let f_1 and f_2 be elements of E^* annihilating P_1 and P_2 , respectively, $\|f_1\| = \|f_2\| = 1$, and such that

$$\begin{aligned}f_1(x) = \rho(x, P_1) &\geq 1 - \gamma_E(\omega) - \varepsilon, \\ f_2(x) = \rho(x, P_2) &\geq 1 - \gamma_E(\omega) - \varepsilon;\end{aligned}$$

then

$$tf_1(x) + (1 - t)f_2(x) \geq 1 - \gamma_E(\omega) - \varepsilon, \quad 0 \leq t \leq 1.$$

Consequently, taking into account that $\theta(L_{f_1}, L_{f_2}) = \theta(P_1, P_2) \geq \omega$, we have

$$\begin{aligned}\beta_{E^*}(\omega) &\leq \max_{0 \leq t \leq 1} (1 - \|tf_1 + (1 - t)f_2\|) \leq \\ &\leq \max_{0 \leq t \leq 1} (1 - |tf_1(x) + (1 - t)f_2(x)|) \leq \gamma_E(\omega) + \varepsilon\end{aligned}$$

and, since $\varepsilon > 0$ is arbitrary, $\beta_{E^*}(\omega) \leq \gamma_E(\omega)$. Finally we obtain $\beta_{E^*}(\omega) = \gamma_E(\omega)$, $0 \leq \omega \leq 1$. Since E is w^* -dense in E^{**} , it follows that $\beta_E(\omega) = \beta_{E^{**}}(\omega)$, and we obtain $\gamma_{E^*}(\omega) = \beta_{E^{**}}(\omega) = \beta_E(\omega)$, $0 \leq \omega \leq 1$. The theorem is proved.

From the facts set forth it follows:

Theorem 3. In order that a Banach space E be uniformly convex (uniformly smooth), it is necessary and sufficient that $\beta_E(\omega) > 0$ (respectively $\gamma_E(\omega) > 0$), $0 < \omega \leq 1$.

Theorem 4. For arbitrary $\varepsilon > 0$ and elements $x \in E$, $y \in E$, $\|x\| = 1$, there is an $\eta = \eta(\varepsilon) > 0$ such that, if $\rho(x, L_y) > 1 - \eta$, then

$$\|x + y\| > 1 + \delta_E \left(\|y\| / \sqrt{2(1 + \|y\|^2)} \right) - \varepsilon.$$

With the aid of Theorem 4 one can obtain a generalization of a theorem of Krein-Krasnosel'skii-Milman on the opening of subspaces ⁽³⁾.

Theorem 5. If P and Q are infinite-dimensional subspaces of a Banach space and $\theta(P, Q) < \frac{1}{2}(1 + \delta_Q(\omega_0))$, where ω_0 is an absolute constant, then $\dim P \geq \dim Q$. As a value, certainly not the best possible, of ω_0 one may take $1/7$.

Theorem 6. For given $\varepsilon > 0$ and elements x, y of a Banach space E , $\|x\| = 1$, there exist an element $z \in E$ and a number τ such that

$$\|x - z\| < \varepsilon, \quad \|y - \tau z\| \leq (1 - \gamma_E(\varepsilon))\|y\|.$$

Definition. We shall say that a sequence $\{e_k\}_{k=1}^\infty$ in a Banach space E has the property of **instability of completeness** in E with respect to the given positive sequence $\{\varepsilon_k\}_{k=1}^\infty$, if there exists a complete sequence $\{g_k\}_{k=1}^\infty$ in E for which $\|e_k - g_k\| < \varepsilon_k$, $k = 1, 2, \dots$

Theorem 7. Let E be a separable Banach space.

If the positive sequence $\{\varepsilon_k\}_{k=1}^\infty$ is such that $\sum_{k=1}^\infty \gamma_E(\varepsilon_k) = \infty$, then every normalized sequence in E has the property of instability of completeness in E with respect to $\{\varepsilon_k\}_{k=1}^\infty$.

In the proof Theorem 6 is essentially used.

Theorem 8. Let E be one of the spaces L_p, l_p , $1 < p < \infty$. In order that an arbitrary normalized sequence in E have the property of instability of completeness in E with respect to a given positive sequence $\{\varepsilon_k\}_{k=1}^\infty$, it is necessary that $\sum_{k=1}^\infty \varepsilon_k^s = \infty$, where

$$s = \begin{cases} p/(p-1), & \text{for } p > 2, \\ 2, & \text{for } 1 < p \leq 2, \end{cases}$$

and it is sufficient that $\sum_{k=1}^\infty \varepsilon_k^r = \infty$, where

$$r = \begin{cases} 2, & \text{for } p > 2, \\ p/(p-1), & \text{for } 1 < p \leq 2. \end{cases}$$

The proof uses Theorems 1, 2, 7, results on the stability of defective unconditional bases ⁽⁴⁾, and the estimate for the modulus of smoothness $\gamma_{L_p}(\omega)$, follow-

ing from the estimate for the modulus of convexity $\delta_{L_q}(\omega)$, $q = p/(p-1)$ (see, for example, ⁽⁵⁾).

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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